

University of Warwick institutional repository: <http://go.warwick.ac.uk/wrap>

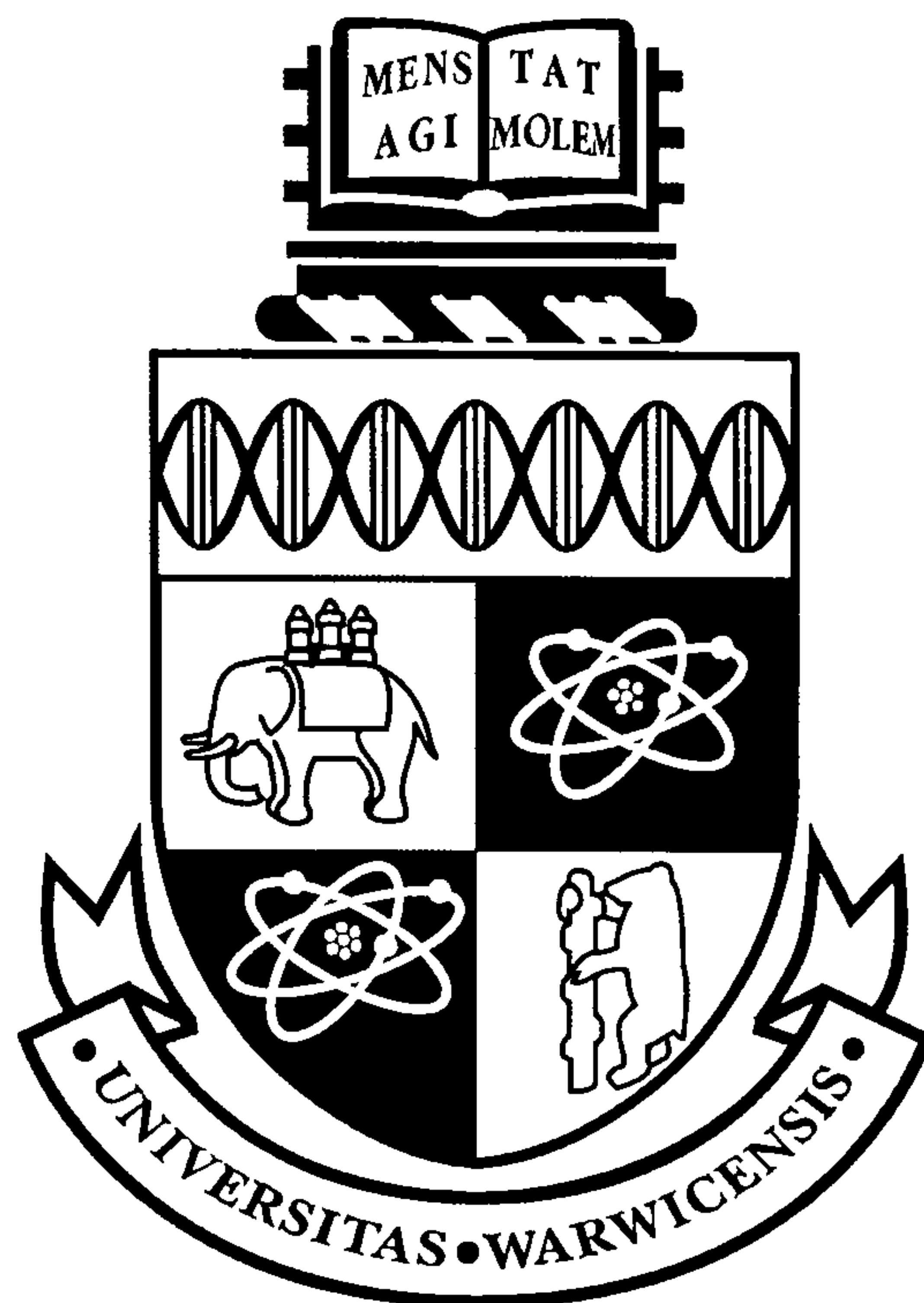
A Thesis Submitted for the Degree of PhD at the University of Warwick

<http://go.warwick.ac.uk/wrap/4063>

This thesis is made available online and is protected by original copyright.

Please scroll down to view the document itself.

Please refer to the repository record for this item for information to help you to cite it. Our policy information is available from the repository home page.



Models for the Price of a Storable Commodity

by

Diana Ribeiro

Thesis

Submitted in partial fulfilment of the requirements to the

University of Warwick

for the degree of

Doctor of Philosophy

Financial Options Research Centre, Warwick Business School

May 2004

THE UNIVERSITY OF
WARWICK

Contents

List of Tables	v
List of Figures	vii
Acknowledgments	x
Declarations	xii
Abstract	xiv
Abbreviations	xvi
Chapter 1 Introduction	1
1.1 Background	1
1.2 Summary of Current Research	5
1.3 Objectives	8
1.4 Overview	9
Chapter 2 Literature Review	13
2.1 Introduction	13
2.2 The theory of Storage	14
2.2.1 Convenience Yield	14

2.2.2	Risk-Premium	18
2.3	Commodity Price Models	19
2.3.1	Structural Models	20
2.3.2	A Partial Equilibrium Real Options Model	41
2.3.3	Reduced Form Models	46
2.4	Conclusion	65

Chapter 3 A Structural Model for the Price Dynamics: Competitive and Monopolistic Markets 70

3.1	Introduction	70
3.2	Storage Equilibrium	75
3.2.1	Model Formulation	77
3.2.2	Linear Inverse Demand Function	85
3.3	Numerical Implementation	86
3.3.1	Space-time Discretization and the Explicit Solution	87
3.4	Results	90
3.4.1	Competitive Case	92
3.4.2	Monopolistic Case	96
3.4.3	Comparison Between the Competitive and the Monopolistic Markets	96
3.5	Extension of the Model to Include Seasonality	100
3.6	Conclusion	101

Chapter 4 A Structural Model for the Price Dynamics: Analysis of the Forward Curve 105

4.1	Introduction	105
4.2	Description of the Lattice Model	108

4.2.1	The Branching Process	108
4.2.2	The Merging Process	113
4.3	Calculation of the Forward Curve and the Convenience Yield . . .	114
4.3.1	Results	115
4.4	Conclusion	124
Chapter 5 A Contango Constrained Reduced Form Model		126
5.1	Introduction	126
5.2	Model Definition	129
5.3	Numerical Implementation	134
5.3.1	Lattice Description	135
5.4	Numerical Results	138
5.4.1	Results from the Lattice Model	138
5.5	Conclusion	146
Chapter 6 A Two-Factor Model for Commodity Prices and Futures		
Valuation		149
6.1	Introduction	149
6.2	Valuation Model	153
6.3	Empirical Estimation of the Joint Stochastic Process	160
6.3.1	State Space Formulation	162
6.3.2	Empirical Results	165
6.4	Conclusion	171
Chapter 7 Conclusion		173
7.1	Summary and Contributions	173
7.2	Limitations and Further Research	179

Appendix A	Derivation of the Stochastic Dynamic Programming Equation	182
Appendix B	Calculation of the Spot Prices Variability	185
Appendix C	Lattice Model for the Ornstein-Uhlenbeck Process	187
Appendix D	Shepard Local Interpolation	191
Appendix E	Derivation of the Process Followed by $x_t = \ln(p_t)$	193
Appendix F	Derivation of the Futures Partial Differential Equation	195
Appendix G	Derivation of the ODEs and corresponding solution	197
G.1	Derivation of the ODEs	197
G.2	Solution to the system of differential equations (6.14) and (6.15) with initial conditions (6.16)	198
Appendix H	Kalman Filter Procedure	201

List of Tables

3.1	Value of the Parameters used to obtain the numerical solutions of both competitive and monopolistic markets.	91
3.2	z_{Min} and z_{Max} represent the lower and upper values of the grid for the exogenous supply rate, z . s_{Max} represents the storage capacity.	91
4.1	T represents the maximum time to maturity considered, dt represents the time-step, l represents maximum number of combinations (s, z) allowed at each node of the tree, and l_{new} is the new number of (s, z) combinations after the merging takes place.	116
4.2	Value of the parameters used to implement the tree for both competitive and monopolistic markets.	116
4.3	z_{Min} and z_{Max} represent the lower and upper values of the grid for the exogenous supply rate, z . s_{Max} represents the storage capacity.	116
5.1	Values of the parameters used in the lattice computation.	138
5.2	Sample moments at final $T = 5$ for the log price sample paths x_T .	145
6.1	Light Crude Oil Futures weekly data from 17 th of March 1999 to 24 th of December 2003.	165

- 6.2 Estimation results and standard errors (in parentheses) for both
our model and Schwartz (1997) [82] two-factor model using all the
futures contracts F0, F1, F2, F3, F4, F5 and F6 from 17th of March
1999 to 24th of December 2003. 167
- 6.3 Summary statistics both our model's and Schwartz (1997) [82] two-
factor model's pricing errors in valuing futures contracts during the
whole period 17th of March 1999 to 24th of December 2003. . . . 168
- 6.4 Market, our model and Schwartz (1997) [82] Two-Factor Model
implied volatilities of annualized log-returns of futures prices. . . . 170

List of Figures

3.1	Competitive Case - Storage rate, u , as a function of the two state variables inventory level, s , and exogenous rate of supply, z	94
3.2	Competitive Case - Super-position of two graphs for the prices in the absence and in the presence of storage, respectively. The price is represented as a function of the two state variables inventory level, s , and exogenous rate of supply, z	94
3.3	Competitive Case - Difference between prices in the presence of storage and prices in the absence of storage as a function of supply rate, at different fixed levels of inventory.	95
3.4	Competitive Case - Price variability as a function of supply rate, at different fixed levels of inventory.	95
3.5	Monopolistic Case - Storage rate, u , as a function of the two state variables inventory level, s , and exogenous rate of supply, z	98
3.6	Monopolistic Case - Super-position of two graphs for the prices in the absence and in the presence of storage, respectively. The price is represented as a function of the two state variables inventory level, s , and exogenous rate of supply, z	98

3.7 Monopolistic Case - Difference between prices in the presence of storage and prices in the absence of storage as a function of supply rate, at different fixed levels of inventory. 99

3.8 Monopolistic Case - Price variability as a function of supply rate, at different fixed levels of inventory. 99

4.1 Part of the tree for computing the forward curve. 112

4.2 Competitive case: Evolution of the forward curve when the initial supply rate is 4.5 and the initial inventory is empty. 120

4.3 Competitive case: Evolution of the convenience yield curve when the initial supply rate is 4.5 and the initial inventory is empty. . . 120

4.4 Monopolistic case: Evolution of the forward curve when the initial supply rate is 4.5 and the initial inventory is empty. 121

4.5 Monopolistic case: Evolution of the convenience yield curve when the initial supply rate is 4.5 and the initial inventory is empty. . . 121

4.6 Competitive case: Evolution of the forward curve when the initial supply rate is 4.5 and the initial inventory is equal to 1.125, 2.25 and 4.5, respectively. 122

4.7 Competitive case: Evolution of the convenience yield curve when the initial supply rate is 4.5 and the initial inventory is equal to 1.125, 2.25 and 4.5, respectively. 122

4.8 Monopolistic case: Evolution of the forward curve when the initial supply rate is 4.5 and the initial inventory is equal to 1.125, 2.25 and 4.5, respectively. 123

4.9 Monopolistic case: Evolution of the convenience yield curve when the initial supply rate is 4.5 and the initial inventory is equal to 1.125, 2.25 and 4.5, respectively. 123

5.1	Forward curves generated by the one-factor mean-reverting model at different spot price levels for a 5-year period.	143
5.2	Forward curves generated by our model at different spot price levels for a 5-year period.	143
5.3	Convenience yield term structures generated by the one-factor mean-reverting model at different spot price levels for a 5-year period. .	144
5.4	Convenience yield term structures generated by our model at different spot price levels for a 5-year period.	144
5.5	Probability density function for the spot prices sample paths at final time $T = 5$ for the unconstrained model.	145
5.6	Probability density function for the spot prices sample paths at final time $T = 5$ for our model.	146
6.1	This figure illustrates the evolution of the forward curve for the market of futures prices and both our model and the Schwartz (1997) [82] two-factor model on the 5th of November 2002. . . .	169
6.2	This figure illustrates the evolution of the forward curve for the market of futures prices and both our model and the Schwartz (1997) [82] two-factor model on the 18th of July 2001.	169
6.3	This figure illustrates the annualized volatility of futures returns implied by our model, Schwartz (1997) [82] Two-Factor Model and the market data.	170
C.1	Alternative Branching at Nodes in a Trinomial Tree	189

Acknowledgments

Having the opportunity of doing a PhD is something I will always be thankful for in life. So many conditions and so many people are required to produce a single thesis: scientific supervision, financial support, friendship and love. Having all these conditions during four consecutive years in my life is something that makes me feel privileged. Doing a PhD is not only about science but it is also about realizing how many people are important in your life and without whom the endeavour of a PhD would not ever be possible.

First and foremost I thank my supervisor, Prof. Stewart Hodges, for believing in this project since my first approach. His comments, encouragement and enduring interest were vital. I would also like to thank my second supervisor, Dr. Elizabeth Whalley, for her valuable comments and support.

I am grateful to *Fundação para a Ciência e Tecnologia*, Portugal for their financial support, without which this PhD would not happen. I also thank the Warwick Business School for their partial support during this last academic year.

I would like to thank my friends in Warwick and in Portugal: thank you for supporting me and listening to me regardless my PhD moods: Carmen, Catarina, Claudia (Ribeiro e Simoes), Cristiana, Elif, Luis Filipe, Nuno, Patricia, Rafael, Rui (Magalhaes e Vieira), Rita, Srini and Susana.

There is one person to whom this thesis also belongs: Arjen. Thank you for your love and immeasurable patience, for your support and belief in me. Thank

you for sharing all the good and the bad moments. You rock my world. Finally, I would like to thank my parents for their endless love and support. Thank you for showing me since early days that knowledge and education are irreplaceable.

Diana

Coventry, May 2004.

Declarations

This is to declare that:

- I am responsible for the work submitted in this thesis.
 - This work has been written by me.
 - This work contains no material which has been accepted for any other degree or diploma in any university or any other institution.
 - To the best of my knowledge and belief, this work contains no material previously published or written by another person, except where due reference has been made in the text.
 - During the preparation of this thesis a number of papers were prepared as listed below. The remaining parts of the thesis are unpublished.
1. Ribeiro, Diana and Stewart Hodges *Price Dynamics for Continuously Produced Storable Commodities: Competitive and Monopolistic Markets*, 8th Portuguese Society for Economic Research Annual Meeting (Lisbon, Portugal), 16-17 June, 2003.
 2. Ribeiro, Diana and Stewart Hodges *Price Dynamics for Continuously Produced Storable Commodities: Competitive and Monopolistic Markets*, The International Society for Intercommunication of New Ideas - Seventh International Congress (Lille, France), 20-23 August, 2003.

3. Ribeiro, Diana and Stewart Hodges *Price Dynamics for Continuously Produced Storable Commodities: Competitive and Monopolistic Markets*, Northern Finance Association Annual Meeting 2003 (Quebec City, Canada), 19-21 September 2003.
4. Ribeiro, Diana and Stewart Hodges *Price Dynamics for Continuously Produced Storable Commodities: Competitive and Monopolistic Markets*, German Finance Association Annual Meeting 2003 (Mainz, Germany), 10-11 October, 2003.
5. Ribeiro, Diana and Stewart Hodges *A Two-factor Model for Commodity Prices and Futures Valuation*, European Financial Management Association 2004 Meeting, (Basel, Switzerland) June 30-July 3, 2004.
6. Ribeiro, Diana and Stewart Hodges *A Two-factor Model for Commodity Prices and Futures Valuation*, Bachelier Finance Society, Third World Congress, (Chicago, Illinois, USA), July 21-24, 2004.

Abstract

The current literature does not provide efficient models for commodity prices and futures valuation. This inadequacy is partly due to the fact that the two main streams of the literature - structural models and reduced form models - are largely disjoint. In particular, existing structural models are developed under rigid discrete time framework that does not take into account the mean-reverting properties of commodity prices. Furthermore, most of the literature within this class does not analyze the properties of the futures prices. Current reduced-form models allow cash-and-carry arbitrage possibilities and do not take into account the dependence between the spot price volatility and the inventory levels.

This thesis investigates three new models for the price of a storable commodity and futures valuation. Specifically, we develop a structural model and two reduced-form models. In doing so, we expand the leading models within each of the two streams of the literature, by establishing a link between them. Each of these models provide an advance of their type.

This study makes several contributions to the literature. We provide a new structural model in continuous time that takes into account the mean reversion of commodity prices. This model is formulated as a stochastic dynamic control problem. The formulation provided is flexible and can easily be extended to encompass alternative microeconomic specifications of the market. The results provide an optimal storage policy, the equilibrium prices and the spot price variability. We also develop a numerical method that allows the construction and analysis of the forward curves implied by this model. We provide a separate analysis considering a competitive storage and considering a monopolistic storage. The results are consistent with the theory of storage. Furthermore, the comparison between monopoly and competition confirm the economic theory. We developed a simple reduced-form model that focuses both on the mean reverting properties of commodity prices and excludes cash-and-carry arbitrage possibilities. This model is compared with a standard single-factor model in the literature. This new model adds two important features to the standard model and motivates the development of a more sophisticated reduced-form model. Accordingly, the last model developed in this thesis is a reduced-form model. It is a two-factor model that represents the spot price and the convenience yield as two correlated stochastic factors. This model excludes cash-and-carry arbitrage possibilities and takes into account the relationship between

the spot price volatility and the inventory level. We find an analytical solution for the futures prices. This model is tested empirically using crude oil futures data and it is compared with one of the leading models in the literature. Both models are calibrated using Kalman filter techniques. The empirical results suggest that both models need to be improved in order to better fit the long-term volatility structure of futures contracts.

Abbreviations

CIR Cox, Ingersoll and Ross

COMEX Commodity Exchange of New York

GBM Geometric Brownian Motion

HJM Heath, Jarrow and Morton

i.i.d. independent and identically distributed

NPV Net Present Value

NYMEX New York Mercantile Exchange

ODE Ordinary Differential Equation

PDE Partial Differential Equation

SDE Stochastic Differential Equation

SREE Stationary Rational Expectations Equilibrium

Chapter 1

Introduction

1.1 Background

Over the past two decades energy markets such as electricity, natural gas, petroleum products and coal have undergone significant changes. The market has evolved from a monopolistic, stable pricing environment characterized by long term contracts with guaranteed margins to a competitive and volatile market environment. In this deregulated environment, market participants have found themselves increasingly exposed to price movements and to counterparty performance risk. Consequently, we have witnessed a rapid expansion in the volume and variety in energy derivatives. This created a need for new analytical tools to accurately price energy contingent claims. From this point of view it should be taken into account that the ability to value financial contingent claims on a commodity significantly depends on the suitability of the models used to replicate the stochastic behaviour of commodity spot and futures prices. Commodity prices in general are harder to model than other well developed conventional financial assets, such as equities. This is partly due to the fundamental price drivers in commodity markets which are more complex than in standard financial assets. In the case of energy commodities

this difficulty is reinforced by the recent dramatic changes in the way energy is traded.

In order to model the behaviour of commodity prices, it is necessary to take into account several key factors that distinguish the stochastic behaviour of commodity prices from conventional financial assets. These key factors are the mean reversion, the stochastic and seasonal demand/supply, and the existence of storage and convenience yield.

Mean Reversion

Generally speaking, mean reversion is the name given to a process by which a variable tends to return to a mean or average value after reaching extremes. Mean reversion in commodity prices appears to be directly related to reactions to market events, which causes imbalances between demand and supply. The uncertainty of the events that may affect commodity prices - such as weather conditions, technical disruptions, wars - are ultimately translated into the uncertainty of commodity prices. Either a correction on the supply side, to match the demand side, or the actual dissipation of the event tends to cause the commodity prices to come back to their typical levels. Thus, strength of the mean reversion in commodity prices is much stronger than the one observed in standard financial markets. Within commodity markets, energy market prices present the strongest mean reversion given the sensitivity of energy markets to demand and supply drivers and the complexity involved in the production, trade and distribution of the finished good.

Stochastic and Seasonal Demand/Supply

Stochastic and seasonal demand and supply are also vital factors in the determination of commodity spot and futures prices behaviour. In agricultural products,

the main source of uncertainty comes from the supply side since the success of the crops is inevitably affected by the weather and soil conditions. Additionally, the production is also seasonal for most of the products. In energy markets such as electricity and natural gas, the primary cause of price uncertainty is caused by the demand. This uncertainty is mostly generated by residential users whereby electricity and gas is primarily used for heating/cooling purposes. In particular, residential users create seasonal effects on electricity and natural gas markets. For example, the United Kingdom consumes natural gas mostly during the winter. Hence, natural gas prices tend to peak mostly during winter and then drop during the summer.

Storage

Another key factor that should be taken into account is storage, which plays a central role in shaping the behaviour of the prices of a storable commodity. On the supply side, storage plays a vital role in stabilizing spot prices by allowing an intertemporal shift of supply in response to shortage. As such, storage is one of the key elements that determines the degree of volatility in commodity prices, that is, the variance of spot price movements over time decreases with the amount in store and vice-versa. In addition, storage limitations may significantly increase the volatility of spot prices when there is a shortfall in commodity's availability. Similarly, storage also influences the extent to which the Samuelson (1965) [79] effect is observed in commodity prices behaviour. This effect implies that spot prices have greater volatility than forward prices, and the variation of forward prices is a decreasing function of maturity. This phenomenon is observed in commodity markets because the supply is more elastic in the long-run than it is in the short run, and spot prices therefore may react strongly to new market information. Nevertheless, these effects are dissipated in the long-run, since the market progressively

adjusts to the new conditions. In other words, storage allows an inter-temporal shift of supply in response to relative scarcity, which implies that the greater the commodity stock, the smaller is the Samuelson effect and vice-versa. It is important to mention that storage has a noticeably asymmetric effect on price. Without storage the probability distribution of a commodity's price is more or less symmetric, with storage it is skewed towards high prices. Although private storers are motivated by expectations of a large supply shortage, in practice they store when there is a surplus. It is assumed that the storage market can always store and storage cannot be borrowed from the future. This implies that storage is more effective in increasing low prices than in moderating sharp price rises. It should be noted, however, that electricity is not storable (in the usual sense), which implies that its behaviour is significantly distinct from other energy commodities. This particularly causes electricity prices to be significantly more variable than oil and natural gas.

Convenience Yield

The existence of a convenience yield in commodity markets is an immediate consequence of the existence of the possibility of storage in the economy. Due to the existence of storage, the relationship between spot prices and forward prices is not only explained in terms of storage costs and the interest rate, but also by the convenience yield. Commodities are frequently stored when the expected price change indicated by futures prices does not cover the time-value of money plus storage expenses. This means that the anticipated revenue from holding inventories is negative, apparently violating arbitrage-free conditions. In those circumstances, producers or consumers benefit from holding the physical commodity instead of holding a contract for future delivery. These benefits, known as the "convenience

yield” are described by Brennan (1991) [9] as the ”flow of services which accrues to the owner of a physical inventory but not to the owner of a contract for future delivery”. The convenience yield reflects the market expectations concerning the future availability of the commodity and therefore declines convexly as the level of inventory increases. Furthermore, seasonality in production or in demand can generate seasonality in inventories and convenience yields.

In summary, commodity prices differ from non-physical financial assets because of the impact of supply and demand conditions as well as the actual consumption of the goods which both dramatically influence the behaviour of commodity prices. In addition, the existence of storage and the convenience yield radically split conventional commodity price models from conventional price models. The accuracy of the valuation of financial contracts contingent on commodity prices ultimately depends on the model assumed for the spot price behaviour. Supply and demand effects converge in the spot market prices and all derivative contracts foresee this convergence. A full understanding of the market behaviour of commodity spot prices furnishes the means for valuing and managing energy derivatives.

1.2 Summary of Current Research

Commodity price dynamics and the economics of commodity storage has been the subject of numerous recent studies. Nevertheless, the research has been largely disjoint. On one hand we assist to the development of discrete time structural models that focus on the behaviour of agricultural commodities and where storage takes a central role in the modelling process. On the other hand, we assist to the recent development of reduced form models that emerged since the energy market became deregulated.

Structural models aim to replicate the equilibrium price for storable com-

modities in a competitive market. All the predominant research focuses on a discrete time framework and is mostly dedicated to the study of agricultural commodity prices, whereby supply is determined by speculative storage and the random behaviour of harvests. The equilibrium price is the solution to functional equations and it is obtained through numerical approximations. The non-negative constraint in storage leads to non-linearity in the models which is carried into non-linearity of the commodity market prices. This stream of literature does not study derivatives pricing but focuses on the empirical behaviour of equilibrium spot and forward prices. The central point of analysis in these studies is the economics of storage and how this affects the commodity spot price behaviour. Early studies include the work of Gustafson (1958a, b) [43, 44], Samuelson (1971) [80] and Lucas and Prescott (1971) [77]. More recently, leading works include Williams and Wright (1991) [89], Deaton and Laroque (1992, 1996) [24, 25], Chambers and Bailey (1996) [12], Routledge, Seppi and Spatt (2000) [78]. The development of structural models is crucial to understand the underpinning interplay between supply, demand and storage in a commodity market and the consequent price dynamics. However, the dependence of the models on non-observable state variables (such as inventory level, random demand shocks, etc) presents considerable practical difficulties for estimation. Moreover, equilibrium prices result from numerical approximations of function relating price supply and storage and therefore do not have closed form analytical solutions. These difficulties make this approach unappealing in practise. One of the gaps in this stream of the literature is that none of the existing studies takes the mean reversion property of commodity prices into account. As mentioned above, it is empirically acknowledged that most commodity prices have mean reversion characteristics. In particular, energy products present a very strong mean reversion.

The recent development of traded options on energy has stimulated the development of reduced form models of energy spot prices. In essence, these models are adapted from more conventional financial assets such as interest rate models. In general, the models with more than one factor treat the spot price and the convenience yield as separate factors with constant correlation. The central focus of these models is to reproduce the mean reversion observed in commodity prices. This property is described either via the spot price process in the case of one factor models, or via the convenience yield process in the case of two and three factor models. Three factor models consider a stochastic interest rate, which is modelled as a third factor. Currently, the most influential two factor models are described by Gibson and Schwartz (1990) [42] and Schwartz (1997) [82]. Three factor models are presented by Schwartz (1997) [82], Miltersen and Schwartz (1998) [68] and Hilliard and Reis (1998) [51]. However, the inclusion of a stochastic interest rate does not improve the valuation of futures prices in comparison with two factor models.

The above described perspective is a standard procedure adopted in both the literature and current practise in energy commodity options pricing. What makes these models attractive is the high analytical tractability they provide, which in turn allows an easy calibration and empirical tests of the models to real market data. Although these models provide powerful tools for derivative pricing and hedging, they also present some problems. First, these models may generate inconsistencies between spot prices and the convenience yield since they are modelled as separate factors. In addition, the magnitude of convenience yield levels is not constrained when prices are in contango thus, these model do not preclude cash-and-carry arbitrage possibilities. Moreover, these models appear to be alienated from the structural models and the theory of storage literatures. In particular,

both spot price and convenience yield have constant volatility and constant correlation between them and therefore do not allow the variance of the spot and forward prices, and the correlation between them, to depend on the inventory level. These inadequacies result in errors in options pricing and are pointed out by Pirrong (1998) [75] and Les Clewlow and Strickland (2000) [16], in particular for long-term maturities and when the market conditions change.

1.3 Objectives

This research develops new models in continuous time for the price of a storable commodity and futures valuation. One of the main goals pursued in this study is to establish a link between two apparently disjoint streams of the literature in commodity price models - the structural models and the reduced form models. We describe two structural models and two reduced form models where each of these represents an advance of their type.

First we develop a structural model with the aim to understand the dynamic interplay between demand, supply and storage in commodity markets. Simultaneously, we incorporate the mean reversion property of commodity prices. As mentioned above, this is a central characteristic of the existing reduced form models in the literature. The structural model presented in this thesis is developed under a general framework, which provides two distinct forms. Each of these forms represents the alternative economic scenarios of competitive and monopolistic storage.

The reduced form models presented in this thesis attempt to improve the leading current models. This is achieved (i) by ensuring that these models exclude cash-and-carry arbitrage possibilities and (ii) by incorporating key properties in commodity price dynamics learned from the analysis of the structural models developed earlier in this thesis and predicted by the theory of storage. In particu-

lar, we take into account the dependency between the spot price volatility and the supply, demand and inventory conditions in the market.

All the models investigated in this thesis have their own important theoretical aspects. The three models investigated are (i) a stochastic structural model where supply evolves as an exogenous mean reverting stochastic process and demand is deterministic, (ii) a one-factor reduced form model where the spot price is driven by a mean reverting process in the absence of storage and the existence of storage constrains the upward drift in the spot price process, (iii) a two factor model where spot prices and instantaneous convenience yield follow a joint stochastic process with constant correlation in the spirit of Gibson and Schwartz (1990) [42] and Schwartz's (1997) [82]; the volatilities in both stochastic processes are time-varying and are proportional to the square root of the convenience yield value.

1.4 Overview

Following the above objectives of this study, Chapter 2 provides a comprehensive and critical review of the existing literature in commodity prices modelling. We first describe the leading concepts and theories proposed by the theory of storage. This helps us to understand the relationship between the economics of storage and the properties of commodity price dynamics. Next, we review the existing models for the price of a storable commodity. The current literature is classified in two main categories as mentioned above. Therefore, the review is structured accordingly: the first part focuses on the structural models while the second part focuses on the reduced form models.

Chapter 3 presents a new stochastic structural model in continuous time for the price of a storable commodity. We develop this model under a general frame-

work, which allows a separate analysis of the competitive and the monopolistic storage markets. The supply evolves as a mean reverting stochastic process of the Ornstein-Uhlenbeck (O-U) type and the inverse demand function is deterministic. This model establishes a link between the structural models and the reduced form model in the literature. Specifically, we draw on a structural model formulation in the fashion of those originally developed to study agricultural commodities and incorporate two features of the reduced form models. In particular, we consider a continuous time framework instead of the standard discrete time framework in the structural models literature and incorporate mean reversion in the model by defining the supply rate as an O-U process. We apply stochastic dynamic programming in continuous time to obtain numerical solutions for the optimal storage policy and the resulting price dynamics. We provide and compare numerical examples for both the competitive and the monopolistic storage markets. Although our formulation is very general, we compute the numerical illustrations of the model considering a linear inverse demand function for simplicity.

In Chapter 4 we implement and analyze the forward curve corresponding to the stochastic structural commodity price model presented in Chapter 3. We introduce a numerical method to calculate the forward curves. Specifically, this method allows us to use the steady state storage policy developed in Chapter 3 to construct a trinomial tree for the commodity prices and the corresponding forward curve. Specifically, the trinomial tree evolves by computing at each node the optimal combinations of supply and storage values, which are obtained through interpolation of the optimal storage policy. We analyze and compare the different types of forward curves and the resulting convenience yield by varying the initial value of inventory level in the model.

In Chapter 5 we develop a new-reduced form model where the spot price is

driven by a mean reverting stochastic process in the absence of storage and where the possibility of storage constrains the upward drift possible in the spot price. Accordingly, the commodity spot price switches between two distinct processes depending on whether or not inventory is being held. The analysis of this model allows us to study the properties of the simplest mean reverting stochastic model possible which satisfies the cash-and-carry arbitrage free contango constrain. The unconstrained version of this model is equivalent to Schwartz (1997) [82] one-factor model. We illustrate and compare the properties of both our model and Schwartz one-factor model using a trinomial lattice.

Chapter 6 introduces a new reduced form model for commodity spot prices and futures valuation which builds and extends the two factor models developed by Gibson and Schwartz (1990) [42] and Schwartz (1997) [82] and takes into account some of the important properties of the structural model developed in Chapter 3 and 4 and predicted by the theory of storage. We develop a two-factor model where the spot price and instantaneous convenience yield follow a joint stochastic process with constant correlation. This model introduces two significant additions to the existing models: it rules out arbitrage opportunities and it considers time varying spot price volatility and time-varying convenience yield volatility. Namely the spot price follows a Geometric Brownian Motion (GBM) where the convenience yield is treated as an exogenous dividend yield and the volatility is proportional to the square root of the instantaneous convenience yield level. The instantaneous convenience yield follows a Cox-Ingersoll-Ross (CIR) which precludes negative values and makes the volatility proportional to the square-root process of the instantaneous convenience yield level. Non-negativity in the convenience yield ensures that our model is arbitrage free. We obtain a closed form solution for the futures prices of the exponential affine form. Finally we test

empirically both our model and Schwartz's (1997) [82] two factor model using light crude oil futures data from for the period from 17th of March 1999 to 24th of December 2003. Due to the non-observability of the state variables, the linearity of the logarithm of the futures prices in the model's state variables and the Markovian property of these we apply Kalman filter techniques.

Finally, Chapter 7 summarizes the thesis, acknowledges its limitations and provides suggestions for future research.

Chapter 2

Literature Review

2.1 Introduction

In this chapter, we provide a review of the existing literature in commodity price modelling. The current literature can be classified into two main categories. The first considers the conventional theory of storage, which examines the essential background necessary to develop models for the price dynamics of storable commodities. The second focuses on developing models for commodity prices and the implication of these in terms of options and futures valuation. This second category is divided into two main approaches - structural models and reduced form models. Next to two main categories we also describe a partial equilibrium storage model that uses the real option models approach. Although the real options literature is beyond the scope of this thesis, we find the description of this particular model relevant since it provides some similarities with one of the storage models developed in this thesis.

We first describe the leading concepts and theories proposed by the theory of storage. This will help us to understand the economics of storage and the properties of the commodity price dynamics. The present research focuses on

the development of continuous time models for commodity prices and futures valuation. The range of current models within each approach will be critically surveyed, in order to underline their corresponding strengths and weaknesses.

Following the above outline, this chapter is organized as follows. Section 2.2 summarizes the main concepts and standpoints related to the theory of storage. Section 2.3 reviews the current literature on commodity price models, considering separately structural models and reduced form models. In this section we also describe a partial equilibrium real options model. Section 2.4 briefly summarizes the previous section and points out that the current models are disjoint, which leads to important misspecifications. Thereafter we propose how we will develop new models to overcome these problems.

2.2 The theory of Storage

The theory of storage literature is divided into two main categories. One seeks to explain the relation between the contemporaneous spot and futures prices in terms of the convenience yield. The other investigates the existence of a risk premium in futures markets. We briefly outline the current literature within each of these fields of study.

2.2.1 Convenience Yield

The class of literature that examines the convenience yield in commodity markets was introduced by Kaldor (1939) [59], and later developed by Working (1949) [90], Brennan (1958) [8] and Telser (1958) [87]. This strand of literature basically attempts to explain the relation between spot and futures prices in terms of convenience yield. Commodities are often stored during periods in which stor-

age returns negative values - that is, when the expected price change indicated by futures prices does not cover the time-value of money plus storage expenses. In other words, the anticipated revenue from holding inventories is negative, apparently violating arbitrage-free conditions. However, producers or consumers of a commodity must benefit from holding inventory. Brennan (1991) [9] describes this flow of benefits, which "accrues to the owner of a physical inventory but not to the owner of a contract for future delivery" as the convenience yield. These benefits may include the ability to profit from temporary local supply shortages of the commodity through the ownership of the physical commodity. The profit may arise from local price variations or from the ability to keep a production process running. The theory of storage explains the difference between contemporaneous spot and futures prices in terms of storage, interest cost of holding inventories and convenience. Accordingly, the non-arbitrage relationship between spot and forward prices is thus given by:

$$F_{t,T} - W_{t,T} = S_t \exp(r_{t,T} - C_{t,T})(T - t) \quad (2.1)$$

where

- $F_{t,T}$ is the forward price at time t , for payment and delivery of a commodity at time $T > t$;
- S_t is the spot price of the commodity at time t ;
- $W_{t,T}$ is the cost of physically storing the commodity from t to T ;
- $r_{t,T}$ is the yield at time t on a discount bond maturing at T ;
- $C_{t,T}$ is the convenience yield;

The convenience yield reflects the market expectations concerning the future availability of the commodity. Brennan (1958) [8] and Telser (1958) [87] provide de-

tailed studies of the relations between convenience yields and inventories for several agricultural commodities. Namely, they show that the convenience yield is a convex function of the aggregate inventory. In other words, the convenience declines as the level of inventory increases. Moreover, they provide empirical evidence that seasonality in production or in demand can generate seasonality in inventories and convenience yields. Samuelson (1965) [79] describes several hypothesis concerning the relative variation of spot and forward prices. In essence he shows that, if (i) the forward price is the expected spot price and (ii) the spot price is a stationary (mean-reverting) process, then forward prices vary less than spot prices, and the variation of forward prices is a decreasing function of maturity. This is known as the *Samuelson effect*, which explains two issues. First, the spot prices volatility is greater than the forward prices volatility. Second, the variation of forward prices is a decreasing function of maturity. This is observed because the supply is more elastic in the long-run than it is on the short-run, since the market forecasts progressive demand and supply responses to shocks.

French (1986) [40] and Fama and French (1987) [37] examine the hypothesis that the relative variation of spot and forward prices is also a function of inventory. Specifically, they study the futures price behaviour of several commodities, including agricultural commodities, wood products, metals and agricultural products over the period 1965-1984. They illustrate the influence of inventory and demand conditions on the spread between spot and forward prices and on the variances and correlation of commodity spot and forward prices. The interest and storage adjusted spread is given by:

$$z_t = (\ln(F_{t,T} - W_{t,T}) - \ln(S_t))/(T - t) - r_{t,T} = -C_{t,T} \quad (2.2)$$

These authors show that for high inventory levels the spread is small and spot and forward prices exhibit similar variances and are strongly correlated. On the

other hand, they explain that for low levels of inventory, the spread increases and a stockout becomes more likely. Since the inventory provides the link between spot and forward prices, the correlation between spot and forward decreases as the inventory levels decrease. Moreover, they illustrate the fact that the effect of demand and supply shocks on the variability of the basis should be an increasing function of storage costs. In addition, they provide empirical studies to illustrate that seasonal variation in the supply of agricultural commodities generate seasonal variation in the basis. Finally, they show that storage costs are important to determine the magnitude of the seasonal variation in the spot prices of agricultural commodities: seasonal variation in the basis is an increasing function of storage costs.

Fama and French (1988) [38] test the theory of storage for business cycles in metal prices over the sample period 1972-1983 and found empirical evidence consistent with the theory. In the case of metals, however, the price behaviour is generated by general business conditions rather than the harvest seasons observed for agricultural commodities. They also show that positive demand shocks around business peaks reduce metal inventories and generate large convenience yields and backwardation.

Cho and McDougall (1990) [13] test the theory of storage in the crude oil, heating oil, and gasoline futures markets using weekly data over the period over 1985-1989. They found evidence that supports the theory for this sector of the energy market as a whole. Susmel and Thompson (1997) [86] test the theory of storage using natural futures monthly data over the period 1975-1994 and storage capacity data in the U.S. market. In particular they show that there was an increase in volatility, which led to an increase in storage activity and increased inventory levels. They also showed that in turn this created an aggregate increase in storage

capacity.

2.2.2 Risk-Premium

We now briefly describe the literature that analyzes the risk premium in futures prices. As elaborated previously, the study of this topic is outside the scope of this thesis. We therefore only outline some of the leading studies within this stream of literature.

The relationship between the futures prices and the expectation of spot prices in terms of risk-premium is defined as:

$$F_{t,T} + RP_t = E[S_T] \quad (2.3)$$

where RP_t is the risk-premium. The risk premium is unobservable and time dependent and can be either positive or negative.

What the above suggests is that backwardation reflects the existence of a positive risk premium, whereby the futures prices are less than the expected future spot prices. Leading studies within this stream of literature are Cootner (1960) [17], Dusak (1973) [33], Breeden (1980) [7], Hasuka (1984) [49] and Deaves and Krinsky (1995) [26]. This viewpoint was introduced by Keynes (1930) [60] who argues that backwardation should prevail for commodity futures. Cootner (1960) [17] further indicates that contango is observed when the hedgers are net short. Deaves and Krinsky (1995) [26] summarize these two views by saying that "commodity futures conform to *risk premium* if backwardation holds when hedgers are net short, and contango if hedgers are net long". If the risk premium is positive, it is viewed as an investors reward for going short in a futures with closing date at time T . On the other hand, if it is negative it is viewed as an investors reward for going long.

Fama and French (1987) [37] analyze the existence of a risk premium in commodity markets by studying the behaviour of futures using agricultural, wood and animal products as well as metals for the period 1965-1984. However, they found that the evidence was not strong enough to support the existence of nonzero expected premiums. Deaves and Krinsky (1995) [26] present an empirical study using data on feeder cattle, live cattle, live hogs, orange juice and also crude oil, heating oil and lumber over a number of different sample periods over 1970-1984. Based on their findings, with the exception of livestock, the question whether any commodity futures are characterized by consistent risk premiums remains open.

2.3 Commodity Price Models

The commodity price models literature is divided into two main approaches:

- structural models and;
- reduced form models.

The structural models aim to replicate the equilibrium price for storable commodities. They are built on the arbitrage-free model as introduced by Samuelson (1971) [80] and further described in Williams and Wright (1991) [89]. Most of the papers in this category focus on establishing an equilibrium discrete time price model for agricultural commodities where supply is determined by speculative storage and the random behavior of the harvests. This stream of literature does not explore derivatives pricing. What this literature concentrates on is the empirical behavior of the equilibrium spot price and on the analysis of the role of storage on the dynamics of this price. The core of the structural models is the solution to functional equations to derive numerical approximations for the functions relating price, supply and storage.

The reduced form models approach focuses on modelling the stochastic behaviour of the spot prices by using a single diffusion (e.g. Brennan and Schwartz, 1985 [10]; Laughton and Jacoby, 1993 [61] and Cortazar and Schwartz, 1997 [19]) or multiple ones (e.g. Gibson and Schwartz, 1990 [42]; Schwartz, 1997 [82] and Miltersen and Schwartz, 1998 [68]) to model the spot price movement. The crucial characteristics of storable commodity prices is the mean reversion (e.g. Gibson and Schwartz, 1990 [42]; Brennan, 1991 [9]; Bessembinder *et al*, 1995 [5] and Pilipovic, 1997 [74]) and the existence of a convenience yield. Accordingly, reduced form models aim to replicate the mean reversion characteristic of the commodity prices taking into account the existence of an exogenous convenience yield. The models differ from each other both in the number of stochastic factors and in the role of the convenience yield in the spot price process. This approach is standard in both the literature and practise and allows significant analytical tractability to price options and futures contracts.

In the next section, we identify and explore both the leading concepts and theories within each of these categories. The review will be structured accordingly.

2.3.1 Structural Models

In this section we present some of the leading works and concepts within the structural models literature for storable commodities. The aim of this review is to introduce the background, core ideas and methodologies underlying this class of models. This section does not provide an extensive review of the literature in storage economics, which is beyond the scope of this thesis.

The models belonging to this category are developed in a discrete time framework and adopt a microeconomics approach, where the equilibrium price for storable commodities is determined by the supply and demand conditions in the

market. A central feature of these models is the fact that it is impossible for the market as a whole to carry negative storage. This leads to non-linearity of the models which is carried on in non-linearity of the commodity market prices. In other words, the non-negativity of inventories produces an "embedded timing option in the spot prices" (Routledge, Seppi and Spatt, 2000 [78]). We will elaborate on this concept later on this review. Most of the papers in this stream focus on establishing an equilibrium price model for agricultural commodities where the supply is determined by speculative storage and the random behaviour of the harvests. The key equilibrium assumptions within this class of models are as follows:

- Rational Expectations Hypothesis: Citing Deaton and Laroque (1992) [24] it predicts that "prices are formed on the basis of expected future payouts of the assets, including their resale to third parties. In other words, a rational expectations market is an efficient market because prices reflect all the information";
- The information available to the agents at time t is the current harvest and the inventory from the previous period;
- Neither consumers nor speculators have advance information about the harvest, and know only the amount on hand immediately after the harvest;
- Storage is competitive and speculative;

Gustafson (1958a, b)[43, 44] introduced dynamic programming as a method to solve for rational expectations behaviour. Muth (1961) [70] also presents a fundamental study that analyzes the rational expectations hypothesis in an isolated market with a fixed production lag. However, he ignores the non-negativity storage constrain as he focuses the attention on the price expectations rather than

on storage. Samuelson (1971) [80] relates competitive storage behaviour to the planner's problem. He considers a model in which the level of output is a given exogenous random variable. In particular, when the time horizon is finite and the supply shocks are independent and identically distributed (i.i.d.) he shows that competitive storage maximizes the market surplus. Lucas and Prescott (1971) [77] also present a competitive equilibrium model under rational expectations for investment, output and prices where the equilibrium evolves *as if* to maximize the expected present value of social welfare in the form of "consumer surplus". This result is also described by Newberry and Stiglitz (1981) [71]. For an infinite time-horizon, Samuelson (1971) [80] extends this model imposing certain restrictions on the price function (for further discussion see Scheinkman and Schechtman, 1983 [81]). Scheinkman and Schechtman (1983) [81] present an exhaustive analytical analysis of the basic storage model and extend previous work by assuming supply responses to the prices.

Williams and Wright (1991) [89] present an extensive literature survey on commodity markets. Using the analytic background of previous work, they explore and extend storage models with the focus on how and to what extent industry wide storage stabilizes the price of a commodity over time. Although they also cover topics such as the government programmes as price bands, buffer stocks and strategic reserves, it must be noted that this is beyond the scope of this thesis.

The models described by Williams and Wright are built on and use the same economic principles as the basic storage models in Samuelson (1971) [80] and Scheinkman and Schechtman (1983) [81]. They describe a multi-period competitive storage equilibrium in two different ways. Initially, the model is based on the perspective of the behavior of individual price-taking firms resulting in an industry level equilibrium. Later on, the same equilibrium model is deduced from

a social planner perspective as to maximize the consumer surplus. Stochastic programming techniques are applied to obtain a numerical solution.

The Basic Equilibrium Model

In this section, we present a basic equilibrium model described in Williams and Wright (1990) [89], which summarizes and unifies earlier works in the literature, namely Gustafson (1958a, b) [43, 44], Muth (1961) [70], Samuelson (1971) [80] and Scheinkman and Schechtman (1983) [81]. This simple model corresponds to an annual agricultural crop whose planting intensity can be adjusted except for the lag of one period, that is, one crop year. The crop output is subject to uncertain conditions (such as the weather). This uncertainty can be interpreted as a shock to production, which is unknown in advance. The uncertainty in each period is independent from the previous period, therefore the shocks are i.i.d. , that is, pure white noise. The market has three distinct groups:

1. consumers;
2. producers;
3. speculative storers.

All the prices are given and the agents are risk neutral with respect to the income. The producers must plan in advance and the storers aim to profit from their activity hold rational expectations about the price for the next period. The consumers' demand curve for current consumption and the producers' supply curve for planned production are stationary functions throughout all the periods considered. In other words, these functions do not depend on the past levels of consumption or production, and there are no trends. The storage has constant marginal costs over

all the ranges stored. Additional complexities, such as fixed costs, location or capacity constraints are not considered in the basic model. The economic problem is then to allocate the total amount available from the carryin plus new production between current consumption and a carryover to the following year, and simultaneously to select a level of planting for the following year. The carryout and planned production are equilibrium quantities; the market clears each period.

With no possibility of borrowing from the future and forward-looking expectations, stochastic dynamic programming or a related recursive method must be used to deduce the competitive equilibrium. The equilibrium can only be solved analytically in special cases and therefore numerical methods are necessary.

The Basic Model from the perspective of the individual price-taking firms

For an individual firm i in the storage industry, its total physical cost of storing an amount s_t from period t to period $t + 1$ is the simple linear function of the quantity it stores:

$$K^i[s_t] = ks_t \quad (2.4)$$

where $k > 0$ is the constant marginal and average physical cost. All the firms have the same technology, and that technology does not change over time. Hence, the aggregation to the industry level carryout $S_t = \sum_{all i} s_t^i$ is straightforward:

$$K^i[S_t] = kS_t \quad (2.5)$$

The expected income under the risk neutral measure by firm i from storage of quantity s_t^i from period t to period $t + 1$ is, as of period t , the difference between revenue in period $t + 1$ and the cost of purchasing s_t^i in the spot market in period t while covering physical storage costs:

$$E_t[\Pi_{t+1}^i] = E_t[P_{t+1}]s_t^i/(1 + r) - P_t s_t^i - ks_t^i \quad (2.6)$$

where r is the rate of interest per period (assumed constant) and $E_t[P_{t+1}]$ is mean of the distribution as of period t of the prices that firm i anticipates could be realized in period $t + 1$. At period t , P_{t+1} is a random variable. For the price-taking firm, its first-order condition for maximization of expected profits is

$$\partial E_t[\Pi_{t+1}^i]/\partial s_t^i = 0 = E_t[P_{t+1}]/(1 + r) - P_t - k \quad (2.7)$$

An individual firm would want to expand its storage to a huge amount if it concluded that expected price for period $t + 1$ was above P_t by more than its costs of physical storage and interest¹. Also, if $E_t[P_{t+1}]$ were below the current price, allowing for storage costs, and if some stocks were being held, an individual firm might consider storing a negative amount. This could be done by borrowing the commodity from other storers for one period, selling the commodity on the spot market, and arranging for yet another party to deliver the commodity the next period, on its behalf, to the lenders. None of these arbitrage situations are allowed. Consequently, equation (2.7) is more properly described as the condition for market equilibrium rather than the first-order condition for an individual firm's maximization. In equilibrium, the net expected profit from a marginal unit of storage must be zero in all the periods. Profit seeking stockholders eliminate any arbitrage situation by adjusting their stocks. If the average price for next period is too high relative to the current period price, storers buy stocks and thereby collectively raise the current spot price. The extra aggregate stocks come from a reduction in current consumption. If the expected price is below the current spot price, aggregate stocks are zero, so there is nothing to be borrowed. Inter-temporal arbitrage can only run in a forward direction, so when stocks are at zero, a signal to store even less cannot be obeyed.

¹This possibility of seemingless limitless profits for an individual firm is the result of the second order condition not ensuring an internal maximum. The objective function (2.6) is linear.

The possible relationships between the amount of collective storage and the expected net profit can be summarized by the following conditions induced by inter-temporal arbitrage conditions:

$$P_t + k - E_t[P_{t+1}]/(1 + r) = 0, \quad S_t > 0 \quad (2.8)$$

$$P_t + k - E_t[P_{t+1}]/(1 + r) \geq 0, \quad S_t = 0 \quad (2.9)$$

Equations (2.8) and (2.9) represent the central condition for competitive equilibrium with storage and state that the price in the current period should never be below the price expected for next period by more than the storage cost; nor above unless the total amount stored is zero.

It is also necessary to determine the prices and the amounts of storage and consumption in the market.

As in the storage industry, consumers are assumed to be price-takers. The quantity q_t consumed identically by many identical consumers in period t is related to aggregate realized production h_t and storage through the identity:

$$q_t = h_t + S_{t-1} - S_t = A_t - S_t \quad (2.10)$$

where A_t denotes market-wide "availability", that is, the amount available in period t from production or previous storage.

Consumption is related to price via the consumption demand curve, which, written in inverse form, is:

$$P_t = P[q_t], \quad \partial P/\partial q < 0 \quad (2.11)$$

In the absence of inter-temporal variation in supply or demand, there would never be any storage in the model, because the net marginal cost of storage is positive. Here, the authors assume that the random disturbance in supply stems from the weather and is a i.i.d. random variable - v_t - and it is in the form of yield. Thus,

the realized harvest is:

$$h_t = \bar{h}_t(1 + v_t) \quad (2.12)$$

where h_t is the planned aggregate production for period t as of the previous period, that is, $E_{t-1}[h_t]$. In the basic model they assume that \bar{h}_t is constant; there is no supply response.

These relationships for consumption and the harvest complete the model and make it possible to deduce the level of current storage and the current price. The arguments in the conditions for inter-temporal arbitrage in (2.8) and (2.9) can be expanded to:

$$P[A_t - S_t] + k = E_t[P[h_{t+1} + S_t - S_{t+1}]]/(1 + r) = 0, \quad S_t > 0 \quad (2.13)$$

$$P[A_t - S_t] + k > E_t[P[h_{t+1} + S_t - S_{t+1}]]/(1 + r) \geq 0, \quad S_t = 0 \quad (2.14)$$

Here, both h_{t+1} and S_{t+1} are random variables, depending on the realization of v_{t+1} .

Basic Model from a social planner perspective

The social planner perspective is an alternative form of deriving the equilibrium storage model. In this case, the equilibrium evolves as to maximize the consumer surplus.

As mentioned above, the dynamic programming approach to solve for rational expectations equilibrium was introduced by Gustafson (1958a, b) [43, 44]. Lucas and Prescott (1971) [77] present a equilibrium model investment, output, and prices in a competitive industry with a stochastic demand by solving a dynamic programming problem which results from the maximization of "consumer surplus". Samuelson (1971) [80] relates competitive storage behaviour to the planner's problem. Williams and Wright (1991) [89] formulate the maximization problem as follows.

The planner's problem in the current period, period t , is to select the current storage that will maximize the discounted stream of expected future surplus, that is:

Maximize with respect to S_t

$$V_t[A_t] = \sum_{j=t}^T E_t \left[\int_0^{A_j - S_j} P[q] dq - kS_j \right] / (1+r)^{j-t} \quad (2.15)$$

subject to $S_j \geq 0$, where $A_t = h_t + S_{t-1}$ as before. A_t is the amount at hand at time t , which results from the current production, h_t , and previous storage, S_{t-1} . Using the terminology of optimal control, we denominate the storage amount, S_t , as the *decision variable*. The relationship between current storage and current availability - the reduced form equilibrium competitive storage can also be called the *storage or decision rule*. The current availability, A_t , is the *state variable*.

Here, surplus each period is measured by the area under the consumption demand curve. The current decision S_t is the primary decision variable, but all future S 's (namely, S_{t+1} through S_T) must be considered too. The process works by backward induction from the final period.

In the final period it is assumed that the planner recommends the consumption of everything available. In this case, the decision is simple. Whatever is A_T , $S_T = 0$. In period $T - 1$ we have:

Maximize with respect to S_{T-1}

$$V_{T-1} = \int_0^{A_{T-1} - S_{T-1}} P[q] dq - kS_{T-1} + E_{T-1} \left[\int_0^{h_T + S_{T-1}} P[h_T + S_{T-1}] dq \right] / (1+r) \quad (2.16)$$

subject to $S_j \geq 0$.

We then obtain the following first order condition:

$$\partial V_{T-1}/\partial S_{T-1} = -P[A_{T-1}-S_{T-1}]-k+E_{T-1}[P[h_T+S_{T-1}]]/(1+r), \quad S_{T-1} > 0 \quad (2.17)$$

An analogous decision rule is then obtained for the period $T-2$, and so on back to the present period t . The subproblems have the general form:

Maximize with respect to S_j

$$V_j[A_j] = \int_0^{A_j-S_j} P[q]dq - kS_j + E_j [V_{j+1}^*[h_{j+1} + S_j]] / (1+r) \quad (2.18)$$

subject to $S_j \geq 0$.

The expectation on the right-hand side is over h_{j+1} , the only random variable, as of period j , in this expression. V_{j+1}^* represents the discounted present value as of period $j+1$ of social welfare provided the planner selects storage in period $j+1$ through period T optimally. The optimal S_j obtained in (2.18) specifies V_j^* , which can be used the optimal S_{j-1} . This *value function* V^* is central to mathematical representations of dynamic programming algorithms.

The general version of the first order condition for S_{T-1} in equation (2.17) is given by:

$$\partial V_j/\partial S_j = 0 = -P[A_j - S_j] - k + E_j[\partial V_{j+1}/\partial S_j]/(1+r), \quad S_j > 0. \quad (2.19)$$

This marginal condition defining the planner's optimal choice of storage in period j can be rewritten as:

$$\partial V_j/\partial S_j = 0 = -P_j - k + E_j[P_{j+1}]/(1+r), \quad S_j > 0. \quad (2.20)$$

When j equals t , the first period in the sequence t, \dots, j, \dots, T is none other than the equilibrium condition for competitive storage, the "arbitrage equation" (2.8), which corresponds to situations where storage is positive.

The identity between the planner's first-order condition and the arbitrage equation for competitive storage implies that competitive storage must be socially optimal.

The model above can also be generalized to an infinite horizon setting. In period t , if period T is sufficiently far in the future, the influence of the anticipation that storage will stop in period T becomes negligible, and the decision rule becomes time independent².

Extensions to the Basic Model

Williams and Wright (1991) [89] describe possible extensions to the basic model such as introducing seasonality and serial correlation in the random perturbations of the "harvests" in opposition to the i.i.d. case initially assumed. A two-lagged supply model is also discussed.

Deaton and Laroque (1992) [24] apply the stationary rational expectations equilibrium (SREE) competitive storage model to study agricultural commodities and confront theory with empirical evidence. They revise the basic model and additionally state and prove the existence of a SREE under the following assumptions described below. They also state some implications of the model to the price behaviour and provide the conditions under which the price follows a renewal process.

As in the basic model, the supply is random and inelastic. They point out that this set up also allows that is possible that the demand is stochastic, in which case the stochastic supply is interpreted as the difference between the harvest and the stochastic part of the demand function. They assume that:

²A discussion of the different numerical methods to obtain a solution to this equilibrium storage model is presented in Williams and Wright (1991) [89] and will not be discussed here. The properties of the models presented in this book are explored using numerical and empirical examples and comparative statics.

- The random supply variables are i.i.d. and have a compact support with lower bound \underline{z} and upper bound \bar{z} ;
- The inverse demand function is positive and finite;
- The inventory holders have access to a simple constant returns storage technology: one unit of the commodity stored at t yields $(1 - \delta)$ units at $t + 1$;
- Inventory holders are risk-neutral and can borrow and lend from a perfect capital market where the rate of interest is r ;
- Inventories are costly and $\beta(1 - \delta) = (1 - \delta)/(1 + r) < 1$.

The inventory holder which carries an inventory S_t into the next period maximizes its profit given by:

$$[\beta(1 - \delta)E_t p_{t+1} - p_t]S_t \quad (2.21)$$

The profit maximization yields:

$$S_t = 0, \quad \text{if} \quad \beta(1 - \delta)E_t p_{t+1} < p_t, \quad (2.22)$$

$$S_t \geq 0, \quad \text{if} \quad \beta(1 - \delta)E_t p_{t+1} = p_t. \quad (2.23)$$

The equilibrium condition is given by:

$$z_t + (1 - \delta)S_{t-1} - S_t = D(p_t). \quad (2.24)$$

Combining (2.22), (2.23) and (2.24) gives:

$$p_t = f(x_t) = \max[\beta(1 - \delta)E_t f(z_{t+1} + (1 - \delta)S_t), P(x_t)] \quad (2.25)$$

where $x_t = z_t + (1 - \delta)S_{t-1}$ ³ and $S_t = x_t - P^{-1}f(x_t)$.

³Note that x_t are also i.i.d. and lies in the interval $[\underline{z}, \infty[$.

Following these assumptions the authors prove that there exists a unique stationary equilibrium f in the class of non-negative and non-increasing functions, which is the solution to the functional equation:

$$f(x) = \max[\beta(1 - \delta)Efz + (1 - \delta)(x - P^{-1}f(x)), P(x)] \quad (2.26)$$

This implies that there is a (constant) price, $p^* = \beta(1 - \delta)Ef(z)$, which implies that whenever the current price is above p^* , there is no storage and the one-period ahead forecast of price is unrelated to the current price. When prices are below p^* , speculators are holding stocks in the expectation that prices will rise.

Deaton and Laroque confront the basic model with the empirical annual data of several agricultural commodities over the period 1900-1987, where the prices are deflated by the U.S. consumer index. They conclude that the model successfully replicates some of the properties of the data, in particular the heteroscedasticity of the prices. Simulated prices also replicate the skewness and excess kurtosis of many of their empirical set of data.

Nevertheless, the authors surprisingly find that the high degree of autocorrelation as presented in many of the commodity prices analyzed cannot be explained by the basic model. In particular, the model predicts that, in the no-storage case, the one-period-ahead forecast of price is unrelated to the current price, which is when the current price is above p^* . In contrast, the autocorrelation of prices in the data is as high at high prices as it is at low prices. They also conclude that for most of the commodities, the one-period-ahead price expectations are better replicated by a simple first-order linear autoregression than by the predictions of the fitted basic model. Williams and Wright (1991) [89] also attribute the strong correlation in the commodity prices strong series to the presence of storage. This suggests that it is necessary to introduce a more realistic storage model.

Deaton and Laroque (1996) [25] argue that, some of the positive auto-

correlation in the supply shocks must be attributed to the underlying processes of supply and demand and that speculative storage is not sufficient. Accordingly, they extend their previous analysis to allow for time-dependent supply shocks. More specifically, they assume that the supply is generated by a first-order autoregressive process. They analyze agricultural commodity prices annual data using the same database as Deaton and Laroque (1992) [24]. They further conclude that the autocorrelation in prices can be explained almost only by the autocorrelation in the supply shocks, where storage plays a little role. On the other hand, they point out that storage is necessary to explain the heteroscedasticity of the commodity prices. They also conclude that the demand shocks are a more plausible source of price fluctuation than has usually been supposed in the literature. They explain this by saying that the weather-driven harvest processes are unlikely to be highly correlated and their results suggest that the source of autocorrelation in prices should be located in the driving process rather than in the mediation of speculative markets.

Chambers and Bailey (1996) [12] also extend the work of Deaton and Laroque (1992) [24] by allowing temporal dependence in supply shocks. These shocks are modelled as a Markov process that can account for a wide range of fluctuations, in which shocks persist from one period to the next. Similarly to Deaton and Laroque (1996) [25] they also build a model based on the concept of SREE and obtain the existence of a (unique) equilibrium price function for models of time-dependent and periodic disturbances. The most general stochastic process studied here can be expressed by a transition function $Q^s(\omega, \omega')$, which can be interpreted as the conditional probability of next period's disturbance. The subscript s indicates that the distribution could differ across periods. The authors consider two particular cases of disturbances. First, similarly to Deaton and Laroque (1996)

[24] the time dependency of the disturbances is captured by the familiar first-order Markov process in which the realized value of the harvest affects the probability distribution of the next period's harvest. Second, they consider periodic disturbances where the periodicity of the disturbances is captured by allowing the transition functions to differ across periods. In both cases, they prove the existence and uniqueness of a SREE. The authors point out that in the most general case, we would allow the transition function to differ among all the periods, which would be "empirically vacuous". In particular, they test the model with periodic disturbances distribution with monthly price data for agricultural commodities over the period 1960-1993. In this periodic disturbances model, time is divided into epochs, each epoch being composed of a fixed number of primitive time periods. The epochs studied in this model are interpreted as years, which comprises an equal number of time periods (months).

The relaxation of the i.i.d. assumption by allowing temporal dependence of the random supply shocks implies that the threshold price, p^* , is no longer a constant independent of time. On the contrary, it becomes a function of the current, observed harvest.

The evidence provided by the data for the periodic disturbances model proves to be supportive but not conclusive. They suggest that a unique specification for the production and marketing of each individual commodity is necessary in order to obtain a more realistic model.

Pirrong (1998) [75] also presents a SREE competitive storage model with focus on commodities that are continuously produced⁴. These may include goods such as industrial metals or energy products. He extends previous work by considering convex and increasing marginal costs (due to storage capacity constraints) and that the uncertainty emerges from stochastic demand, which includes both perma-

⁴Although his model presents the standard discrete time framework.

nent and persistent shocks. This latter assumption is an extension to the existing literature since previous works only consider either i.i.d. shocks only (Scheinkman and Schechtman, 1983 [81]; Deaton and Laroque, 1992 [24] and most of Williams and Wright, 1991 [89]) or persistent shocks only (Deaton and Laroque, 1996 [25] and Chambers and Bailey (1996) [12]). Pirrong (1998) [75] shows that this model replicates the heteroscedasticity of the spot prices. In particular, these prices are highly volatile for tight supply/demand conditions (high demand and/or low inventory) and exhibit little volatility when supply is abundant (due to low demand and/or high inventories). This study analyzes the relationship between inventory, demand, prices, spot-forward spreads, spot returns variances, and spot-forward return correlations and shows that the properties of the model are consistent with the dynamics of the prices of continuously produced commodities and with the theory of storage. The analysis presented by Pirrong also shows that the traditional commodity option pricing models⁵ do not price options very accurately. Furthermore Pirrong suggests that the direction of these pricing errors depends on the market conditions. In particular, he analyzes European calls and illustrates the mispricings generated by the two-factor model in Schwartz (1997) [82]. These pricing errors are more severe in the case of short-dated out-of-the-money calls than for in-the-money long-dated calls. More specifically, when the demand is high and the inventory is very short, the constant volatility benchmark model severely underprices options. On the other hand, when demand is low and/or inventory is high, the same model overprices the options but not as severely. Pirrong then concludes that the standard reduced form models generate significant errors in pricing options because commodity price volatility is significantly dependent on the supply/demand and storage conditions of the market. This state dependency

⁵These are the reduced form models, which will be revised later in this chapter, for example Gibson and Schwartz (1991) [42], Amin, Ng and Pirrong (1996) [1] and Schwartz (1997) [82].

of commodity prices also implies state dependency in the volatility smiles and hedging ratios. Although this model presents a relevant study in commodity price models, it is not yet tested empirically. The comparisons between Pirrong's model and Schwartz (1997) [82] model would be stronger if compared empirically. Furthermore, although Pirrong claims that his model is designed to take into account high frequency data, there is not a significant difference between his formulation and previous works.

Routledge, Seppi and Spatt (2000) [78] characterize spot and forward commodity prices in a SREE model. Their analysis builds on and extends the studies of Deaton and Laroque (1992, 1996) [24, 25], Chambers and Bailey (1996) [12], Wright and Williams (1989) [91] and Williams and Wright (1991) [89]. Routledge, Seppi and Spatt initially characterize a single-factor model that considers exogenous transitory shocks to supply and demand. Later they extend this model incorporating a second factor with permanent shocks. The most innovative feature in this paper is the detailed analysis of the equilibrium forward curves generated by their model. In particular, the authors focus their analysis on how the exogenous stochastic demand and endogenous storage determine the shape of the equilibrium forward curves.

The single-factor model assumes that the net demand transitory shocks, $a_t \in \Omega$, where Ω is a probability space, are realizations of a finite dimensional, irreducible, m -state Markov process ($m \geq 2$), with transition probabilities $\pi(a|a_t)$ in a matrix Π . The authors assume that $\Pi \gg 0$ (Π is positive definite) which implies that the shocks have a limiting distribution that is independent of the current state. Under standard assumptions they prove the existence of a SREE and derive some properties of the equilibrium inventory process. The equilibrium prices are also given by the central standard condition for competitive equilibrium

with storage given by equations (2.22) and (2.23) similarly to Deaton and Laroque (1992, 1996) [24, 25].

Routledge, Seppi and Spatt classify the net demand shocks into "sell" states and "buy" states. The "sell" states occur at positive levels of storage whenever inventory is not accumulated. In particular, if storage levels are particularly low and a "sell" state occurs, then all inventory is sold for consumption and a "stock-out" occur. However, any time the inventory "stocks out", the inventory process "regenerates" or renews. Using this description, they present a numerical example where they specify a persistent Markov process for the net demand shocks, a_t for $m = 2$. Accordingly, there are only two demand states - "high", a_H , and "low", a_L . According to this designation, they generate a set of forward prices. In particular, they illustrate that, with a low or zero prior inventory, forward curves are upward sloping in the low demand state, a_L and downward sloping in the high demand state, a_H . When demand is high and the incoming inventory is at a moderate level, then the forward curve can be "hump shaped"⁶. More specifically, forward prices initially rise, but eventually decline to a state-independent forward price. In addition, they illustrate the inventory dynamics by generating an endogenous binomial tree of forward curves over time. The starting node corresponds to a stockout and high demand state, which generates a forward curve in backwardation. This tree also illustrates that the degree of backwardation or contango of the forward curve depends on the level of inventory accumulated. Hump shaped curves are also generated in this tree when a contango forward curve ("low" demand state) at time t is followed by a high demand state at $t + 1$. Moreover, the hump moves further back along the forward curve if a series of successive low demand shocks leads to accumulation of inventory pushing the possibility of a stockout further

⁶By "hump shaped" is meant that the forward curve presents initially contango followed by backwardation

into the future.

The convenience yields are endogenously derived from the cost-of-carry relation by opposition to the exogenously defined convenience yields in the standard reduced form models. As a result the convenience is well defined, that is, it is always non-negative and ensures consistency between spot and forward prices, unlike to the reduced form models described later. Here, convenience yields are function of both exogenous demand state and endogenous inventory whereas in the reduced form models it only depends on the spot price. The authors illustrate that the dynamics of the convenience yield are in harmony with the properties stated by the theory of storage. Moreover, they show that backwardation (or positive convenience yields) occurs whenever the current inventory is low or stocked out. This happens regardless the existence of explicit service flows. In the standard stochastic diffusion models (e.g. Schwartz (1997) [82]) spot prices and convenience yields have a constant correlation. In this model this correlation depends on the endogenous inventory level.

They also investigate the volatility structure of the forward prices and show that the "Samuelson Effect"⁷ need not to hold *conditionally* in all states at all horizons. Conditional violations might occur for sufficiently high inventory levels and short time horizons. In particular, when current inventory is high, the spot price is less volatile than the short horizon forward. In addition, the authors relate storage costs with volatilities. Deaton and Laroque (1992, 1996) [24, 25] and Chambers and Bailey (1996) [12] show that the spot price volatility is reduced by the presence of storage. This, in turn, is also reflected in the ability of storage to smooth the forward price volatilities. Because high storage costs deter storage, the smoothing effect of the storage on the forward price volatility is a decreasing

⁷As mentioned before, the *Samuelson effect*, proposed by Samuelson (1965) [79], states that future prices variance decrease with maturity.

function of the storage costs. This result is also analyzed in French (1986) [40] and Fama and French (1987) [37]. Routledge *et al* (2000) [78] calibrate their single-factor model to crude oil futures prices. They specify a power inverse net demand function and use a two-state Markov process to approximate a first-order autoregressive process. However, this model has been unable to capture both the conditional and the conditional volatilities of the data. As a result they extend the single-factor model into a two-factor model that includes both transitory and permanent net demand shocks and calibrate it to the oil futures data. This model performs better in capturing both the unconditional and conditional futures prices volatility empirical data.

The analysis of equilibrium forward curves presented by Routledge Seppi and Spatt is renewing and useful, since none of the existing structural models have analyzed the forward curve. With the exception of Routledge, Seppi and Spatt (2000) [78], all the other equilibrium models focus only on the dynamics of commodity spot prices and the interplay between prices, supply and inventory. Hence, these works do not analyze the forward curves implied by the corresponding models. From this perspective, the analysis of Routledge Seppi and Spatt is very renewing. Nevertheless, this examination is limited since it considers only two possible states for the stochastic demand. More specifically, there are only two demand states - "high", a_H , and "low", a_L . Therefore, the analysis presented is very limited and is difficult to generalize to a more general Markov process.

The advantage of the structural models as described above is that they allow us to define the demand, supply and storage conditions of the market explicitly. This, in turn, enables us to identify, measure and understand the relationship between the fundamental economics of the market and the price dynamics generated by the model. Furthermore, these models calculate the convenience yield

endogenously based on the relationship between futures prices and the corresponding contemporaneous spot prices. This ensures that the convenience yield is well defined and does not generate inconsistencies between spot and forward prices or contango violations of arbitrage-free conditions. Nevertheless, the current research presents a number of drawbacks. First, all the current models are developed under a discrete time framework, since most of them focus on agricultural commodities, which are harvested once a year. As a result, these models are tested using very low frequency data. For example, Deaton and Laroque (1992, 1996) [24, 25] use annual data. Chambers and Bailey (1996) [12] use monthly data since they incorporate the seasonality in their model. The use of such low frequency data and the fact that none of these studies was tested using continuously produced commodities poses the question whether these models are adequate of continuously produced commodities. Examples of these commodities are energy commodities, such as natural gas and oil. It would be more adequate to develop a continuous time framework to study such commodities. Second, the existing studies in the literature do not take into account the mean reversion property of commodity prices. It is empirically acknowledged that most commodity prices have mean reversion characteristics (see, e.g. Bessembinder *et al*, 1995 [5]). On the contrary, mean reversion is the central property of reduced form models in the literature. This indicates that the structural models and the reduced form models literature is largely disjoint. Finally, the other disadvantage of the structural models is related to practical issues. Specifically, structural models do not have the analytical tractability which makes the reduced form models appealing in practice. In particular, the in most of the cases solutions must be obtained applying relatively complex numerical methods, which implies that empirical applications are extremely gruelling.

2.3.2 A Partial Equilibrium Real Options Model

As acknowledged previously, in this section we do not intend to review the real options literature. However, we find it important to present a specific optimal storage policy model described by Fackler and Livingston (2002) [36] that uses the real option valuation approach. The reason why we find this important is because this model combines both the structural models and the reduced form models approach likewise one of the models we develop later on in this thesis. More specifically, this model applies stochastic dynamic programming formulation to obtain a optimal storage policy likewise structural models. Similarly to reduced form models, this model applies a continuous time framework and represents the commodity spot price by a stochastic diffusion.

The valuation approach suggested by the term "real options" applies the options valuation literature to optimal investment decisions. Brennan and Schwartz (1985) [10], McDonald and Siegel (1986) [64], Paddock, Siegel and Smith (1988) [73] and Dixit and Pindyck (1994) [28] established the analogy between investment decisions and the literature on the valuation of financial options. In particular, these authors recognized that making an irreversible investment is equivalent to exercising an American-style option and applied the finance literature to solve for the value of an investment decision.

Facker and Livingston (2002) [36] present a stochastic dynamic programming model in continuous time for the optimal storage policy, developed under simplistic assumptions. Besides that this model considers a continuous time framework, this model differs from the structural models described above in two assumptions. First, in this model it is assumed that the storage decision rule does not affect the spot price. On the contrary, the storage decisions affect the equilibrium prices in structural models. Second, Fackler and Livingston assume that the

stockholder, which is also a producer, is not allowed to buy from any other producers. In this model, the optimal storage rule is reduced to a simple condition on the spot price: sell everything if the current price is high; sell nothing otherwise. Accordingly, there is a cutoff price function that represents the price at time t at which the storer is indifferent between selling and buying. The model concentrates on modelling mean reversion and seasonality in prices, whereby prices follow a O-U modified to allow seasonal variation in the mean and variance. This is appropriate since mean reversion is a well documented property of commodity prices (e.g. Bessembinder *et al*, 1995 [5]; Gibson and Schwartz, 1990 [42] and Brennan, 1991 [9]). Additionally, several commodities exhibit seasonality, such as most of the agricultural commodities and energy. There are two state variables: the exogenous spot price and the inventory level. The exogenous spot prices, p_t , is given by:

$$dp = \alpha(a(t) - p)dt + b(t)dz \quad (2.27)$$

where $a(t)$ and $b(t)$ are seasonal functions, which are defined as periodic functions with periodicity of one year.

The storage level, s is a fully controllable endogenous variable which satisfies the following transition equation:

$$ds = -q(t)dt \quad (2.28)$$

where $q(t)$ represents the rate of sales.

The optimization problem is subject to the constraints that $q \geq 0$ and $s \geq 0$. The first constrain implies that the producer cannot buy from other producers. Also note that there is not an upper bound placed on q . This implies that it is possible for the producer to sell all stocks immediately. In order for this to be possible in a continuous time model, the rate of sales must become infinite at a single instant

in time. The second constrain means that storage is non-negative, which is a basic assumption. Note that there is not a limit on storage capacity.

The producer is a risk neutral price taker. His objective is to maximize the expected present value of wealth over the period of one year, starting from an initial stock of harvested production. The risk free interest rate, r is assumed to be constant. The net income flow at time t , $\pi(t)$ is equal to the proceeds from sales less the costs incurred on the currently held stocks:

$$\pi(t) = q(t)p(t) - ks(t). \quad (2.29)$$

The optimization problem, formulated in terms of the value function, $V(s, p, t)$, is the following:

$$V(s, p, t) = \max_{q, p, t} E \left[\int_t^T e^{-r(l-t)} (q(t)p(t) - ks(t)) dl \right]. \quad (2.30)$$

The value function also satisfies the Bellman (1957) [4] equation:

$$rV = \max_q qp - sk + V_t - qV_s + a(t)V_p + \frac{b(t)^2}{2}V_{pp} \quad (2.31)$$

where the subscripts denote partial derivatives. The Bellman equation holds when stocks are positive, that is, for $s > 0$; when the inventory is empty, that is, when $s = 0$, the only feasible value for the control is $q = 0$ and thus $V(0, p, t) = 0$. The first order conditions associated with the optimization problem are:

$$V_s - p \geq 0, \quad q \geq 0, \quad (2.32)$$

$$q(V_s - p) = 0, \quad \text{for } s > 0. \quad (2.33)$$

Thus in the event that the shadow price of stocks is greater than the market price $V_s > p$, it is optimal to hold all of the stocks ($q = 0$). On the other hand, if the price is high enough to justify the sale of some of the stocks, the linearity

of the profit function in the rate of sales, q , and the lack of an upper bound on q , implies that it would be optimal to sell at an infinite rate, instantly selling all available stocks. The optimal decision rule can be summarized as a hold high, sell low rule.

The difficulty of this problem is to find the price at which one is indifferent between holding and selling stocks, $c(t)$. This requires solving the Bellman equation for low prices, where $q = 0$. Fackler and Livingston show that this problem is equivalent to an optimal stopping problem in Dixit (1993) [27]. The Bellman equation (2.31) for $q = 0$ is also satisfied by the function $sv(p, t)$, where $v(p, t)$ satisfies:

$$rv = -k + v_t + a(t)v_p + \frac{b^2(t)}{2}v_{pp}. \quad (2.34)$$

The optimal decision rule can be determined by the unit value function v and the optimal cut-off price $c(t)$ for $0 \leq p(t) \leq c(t)$ by solving this equation subject to the terminal condition:

$$v(p, T) = p \quad (2.35)$$

and the boundary conditions:

$$v(c(t), t) = c(t) \quad (2.36)$$

$$v_p(c(t), t) = 1. \quad (2.37)$$

The boundary conditions are respectively known, as the value matching and smooth pasting conditions. The authors point out the analogy between this equation and the optimal stopping problem of determining the optimal time to exercise and American option.

Since the solution of this problem is non trivial and requires numerical methods to find the solution, Fackler and Livingston examine two suboptimal rules. The first is myopic rule, which is to hold if the instantaneous rate of change in

the price is greater than the opportunity cost of capital plus the physical storage costs, that is, if:

$$a(t) > rp_t + k \quad (2.38)$$

and to sell otherwise. This leads to the rule where we hold if $p < c^m(t)$, where the myopic cutoff price, $c^m(t)$ satisfies:

$$a(t) - rc^m(t) - k = 0. \quad (2.39)$$

The other suboptimal rule is to hold if the maximum expected return to holding stocks is greater than their current sales value. The time t value of the cost of storing from time t to time $t + h$ is:

$$k \int_0^h e^{-r\tau} d\tau = (1 - e^{-rh}) \frac{k}{r}. \quad (2.40)$$

The expectation rule can therefore be expressed in terms of a cutoff price, that is to hold if $p(t) < c^e(t)$, where $c^e(t)$ is the minimum price such that:

$$0 = \max_{h \in [0, T-t]} e^{-rh} E[p(t+h)|p(t)] - (1 - e^{-rh}) \frac{k}{r} - p(t) \quad (2.41)$$

for any $h \in (0, T - t]$.

The authors show that (i) holding stocks under the myopic rule implies holding stocks under the expectations storage policy and (ii) holding stocks under the expectation rule implies holding stocks under the optimal storage policy. Consequently, the cutoff prices satisfy:

$$c^m(t) \leq c^e(t) \leq c(t). \quad (2.42)$$

This inequalities imply that the suboptimal decision rules may underestimate the optimal cutoff price. The model is calibrated using soybeans weekly prices based on a database that covers the period November 1975-October 1997. The authors show the results for the optimal rule and both the sub-optimal storage rules.

They conclude that the optimal storage policy produces substantial return for stockholders. The application of the suboptimal storage policies generate similar results, although the expectation rule performs better than the myopic rule.

This model produces a useful analysis of a partial equilibrium continuous time storage model that includes key properties of commodity prices such as mean reversion and seasonality. Additionally, it has an elegant formulation, which is similar to optimal stopping problems, particularly real options problems. Nevertheless, this model also presents several limitations. In particular, there are several assumptions in the model that are not realistic: (i) the decision rule does not affect the prices (ii) sales must always be positive, that is, the stockholder is not allowed to buy from other producers and (iii) there is not a limit on storage capacity. These assumptions imply the solution to the stochastic dynamic problem is of the "bang-bang" type. In other words, the stockholder either does not sell at all or he sells everything, depending on the spot price level. This type of storage policy is clearly unrealistic.

2.3.3 Reduced Form Models

Log-normal Spot Price Diffusion Models

In this section, we describe a large and important class of models, which assume that the spot price follows a log-normal diffusion process, which is usually also mean-reverting. The main purpose of this approach is to provide models for derivatives pricing and futures valuation, since reduced form models allow the valuation of contingent claims under the risk-neutral measure.

In essence, the models within this category differ from each other in the way they incorporate the convenience yield in the spot price process and in the number of stochastic factors that contribute for uncertainty.

Single-Factor Models

Brennan and Schwartz (1985) [10] assume that the commodity price follow a geometric Brownian motion (GBM) and that the convenience yield is constant. This model does not take into account the mean reversion property of the spot commodity prices and neglects inventory-dependence property of the convenience yield. Brennan (1991) [9] shows a constant convenience yield assumption only holds under clearly unrealistic assumptions. The Schwartz (1997) [82] model takes into account the mean reverting properties of commodity prices by assuming that the commodity spot price follows the stochastic process:

$$dS = \kappa(\mu - \ln(S))Sdt + \sigma Sdz \quad (2.43)$$

where κ is the speed of mean reversion. By defining $X = \ln S$ and applying Ito's Lemma, the log-price is characterized by an O-U stochastic process:

$$dX = \kappa(\alpha - X)dt + \sigma dz \quad (2.44)$$

$$\alpha = \mu - \frac{\sigma^2}{2\kappa} \quad (2.45)$$

where α is the long run mean log price. Based on this model, the volatility structure of futures prices tends to zero as the maturity increases and the futures prices will converge to a constant as maturity increases. In reality, however, the volatility of forward prices decreases with maturity but does not become zero. This misspecification leads to severe disparities when pricing options on futures contracts (c.f. e.g. Clewlow and Stickland, 2000 [16]). Another examples of single-factor models that take onto account the mean-reverting properties of prices but do not consider a convenience yield can be found in Laughton and Jacoby (1993) [61], Cortazar and Schwartz (1997) [19].

Two-Factor Models

Gibson and Schwartz (1990) [42] and Schwartz (1997) [82] introduce a two-factor, constant volatility model. They assume that the spot price, S , and the net convenience yield, δ , follow a joint diffusion process specified as

$$\frac{dS}{S} = \mu dt + \sigma_1 dz_1, \quad (2.46)$$

$$d\delta = \kappa(\alpha - \delta)dt + \sigma_2 dz_2, \quad (2.47)$$

where dz_1 and dz_2 are correlated increments to standard Brownian processes and $dz_1 dz_2 = \rho dt$, where ρ denotes the correlation coefficient between two Brownian motions. The differences between Gibson and Schwartz (1990) [42] and Schwartz (1997) [82] are the estimation methods employed to calibrate the model and the data set used to empirically test the model.

Gibson and Schwartz used New York Mercantile Exchange (NYMEX) oil futures over the period January 1984 - November 1988. The parameters are estimated using seemingly unrelated regression. The convenience yield is not observable and the spot prices of crude oil are difficult to observe. For this reason, this estimation method calls for the definition of two proxies for the state variables S and δ . The proxy for the spot price is defined by the shortest maturity futures price available at each date of the sample period. The annualized convenience yield is calculated using pairs of adjacent monthly futures contracts. This approximation may affect the empirical test of the model. Gibson and Schwartz (1990) [42] show that the model is reliable for the purpose of valuing short term futures contracts. However, the performance of the model decreases as the futures maturity increases. Therefore the model is not suitable to evaluate long term oil futures contracts.

Schwartz (1997) [82] tests the model empirically using weekly observations of futures prices for two commercial commodities over the period July 1988-June 1995. These are oil and copper, and one precious metal, gold. The parameters are estimated using Kalman filter techniques. The use of this technique is more appropriate to test this model than the method employed by Gibson and Schwartz (1990) [42] since: (i) the state variables are non-observable, (ii) the futures prices valuation formula is linear in the state variables of the model and (iii) the model state variables have a Markovian structure. Harvey (1989) [48] provides a detailed and comprehensive description of the Kalman filter.

The volatility of futures returns implied by the models of Gibson and Schwartz (1990) [42] and Schwartz (1997) [82] decreases with maturity and converges to a positive constant. This is consistent with a mean-reverting non-stationary process and with the futures term structure observed in the market. However these models do not permit the state dependence in the volatilities implied by the theory of storage. It is empirically established that commodity spot prices are heteroscedastic and that the spot price volatility increases when the inventory level decreases (see the review of the theory of storage above). Similarly, the convenience yield increases as the inventory level decreases. This suggests that there is an implicit relationship between the volatility of the spot prices and the instantaneous convenience yield. In summary, models with constant volatility and correlation conflict with the theory of storage and therefore have drawbacks for option pricing as we mention later in this section.

Pilipovic (1997) [74] presents a two-factor mean-reverting model where spot prices revert to a long-term equilibrium level which is itself a random variable:

$$dS_t = \alpha(L_t - S_t)dt + S_t\sigma dz_1 \quad (2.48)$$

$$dL_t = \mu L_t dt + \xi L_t dz_2 \quad (2.49)$$

where dz_1 and dz_2 are two standard Brownian motions, L_t is the long term equilibrium price, α is the spot price rate of mean reversion, σ is the spot price volatility and ξ is the long-term price volatility. As Clewlow and Strickland (2000) [16] point out, Pilipovic derives a closed-form solution for forward prices to her model when the spot and long-term prices are uncorrelated. However she does not discuss option pricing in her two-factor framework. It is therefore unclear what this model adds to the existing literature in terms of commodity futures and option pricing. Schwartz and Smith (2000) [84] present an alternative short-term/long-term model. They decompose the spot prices as $\ln(S_t) = \chi_t + \xi_t$, where χ_t represents the short-term deviation in prices and ξ_t the equilibrium price level. The short-term deviations (χ_t) are assumed to revert toward zero following an O-U stochastic process:

$$d\chi_t = -\kappa\chi_t dt + \sigma_\chi dz_\chi \quad (2.50)$$

and the equilibrium level (ξ_t) is assumed to follow a Brownian motion process

$$d\xi_t = \mu dt + \sigma_\xi dz_\xi \quad (2.51)$$

Here dz_χ and dz_ξ are correlated increments of standard Brownian motion processes with $dz_\chi dz_\xi = \rho_{\chi\xi} dt$. The authors show that although this model does not explicitly consider a convenience yield it is equivalent to the Gibson and Schwartz's (1990) [42] and Schwartz (1997) [82] models in the sense that the factors in each model can be represented as a linear combination of the factors in the other. Schwartz and Smith present a closed-form solution for the futures prices and European options on futures contracts. Likewise the models described above, the logarithm of the spot prices is a linear function of the model state variables. For this reason and because the state variables of the model are Markovian and non-observable, the authors apply the Kalman filter to estimate the parameters of the model. The performance of this model is evaluated using a data set of crude oil

futures contracts with maturities varying from one month to one year. This is the same data set as used in Schwartz (1997) [82]. The resulting pricing errors show that the model is good to evaluate medium term maturity futures contracts. Nevertheless, its performance decreases significantly for the short and long-term contracts. The authors compare their model to two simple single-factor models where the spot price follows a geometric Brownian motion and an Ornstein-Uhlenbeck (O-U) process, respectively. They conclude, not surprisingly, that their model performs better in terms of pricing errors. However, they disregard any empirical comparison between this model and the more advanced models in the contemporaneous literature, such as Gibson and Schwartz (1990) [42] and Schwartz (1997) [82] two-factor model. Accordingly, it is not very clear what the Schwartz and Smith (2000) [84] model adds to the existing models.

Schwartz (1998) [83] presents a single-factor model for commodity prices that has practically the same implications as the two-factor model when applied to evaluate long-term commodity assets but simplifies its evaluation. It is derived directly from the Schwartz (1997) [82] two-factor model by analyzing its behavior when the futures time to maturity goes to infinity. The contribution of this paper consists basically in simplifying the computational implementation of the two-factor models in the literature, which becomes an advantage to evaluate complex contingent claims. However, as pointed out by Clewlow and Strickland (2000) [16], it is precisely in the valuation of complex options that this model produces poor results.

Three-Factor Models

Schwartz (1997) [82] extends the two-factor model to a three-factor model where the third factor is the stochastic interest rate. The interest rate follows an O-U stochastic process as in Vasicek (1977) [88]. Now, the joint stochastic process for

the factors under the equivalent martingale measure are expressed as:

$$\frac{dS}{S} = (r - \delta)dt + \sigma_1 dz_1^* \quad (2.52)$$

$$d\delta = \kappa(\hat{\alpha} - \delta)dt + \sigma_2 dz_2^* \quad (2.53)$$

$$dr = a(m^* - r)dt + \sigma_3 dz_3^* \quad (2.54)$$

$$dz_1^* dz_2^* = \rho_1 dt, \quad dz_2^* dz_3^* = \rho_2 dt, \quad dz_1^* dz_3^* = \rho_3 dt \quad (2.55)$$

where a and m^* are, respectively, the speed of adjustment coefficient and the risk adjusted mean short rate of the interest rate process. This model has a closed form solution for the corresponding futures prices, which is exponential affine in the state variables, similarly to the Gibson and Schwartz (1990) [42] and Schwartz (1997) [82] two-factor model. The parameters are estimated using Kalman filter techniques for the same reasons as mentioned for the two-factor models. The results presented by Schwartz (1997) [82] show that the inclusion of a stochastic interest rate in the model does not improve the two-factor model performance.

Hilliard and Reis (1998) [51] present a three-factor in the spirit of Schwartz (1997) [82] but use a risk-neutralized Heath, Jarrow and Morton (HJM) (1992) [50] no-arbitrage process for the interest rate. This eliminates the market price of interest rate risk in the valuation of futures contracts and options. However adding a stochastic interest rate does not affect the futures and options valuation in practice. Additionally, Hilliard and Reis (1998) [51] further extend this three-factor model by adding jumps to the spot price diffusion process. They show that the inclusion of jumps in the spot price does not affect the futures valuation but can have implications in the options price. This occurs because futures and forwards prices are obtained as expectations over the entire distribution of the of the underlying asset and option price is obtained as an expectation over a portion

of the distribution of the underlying asset. Therefore, non-normal skewness and kurtosis for the underlying asset's returns affect options prices, but not futures and forward prices. However, this assumption implies loss of the analytical tractability and therefore requires a numerical solution to option pricing problems. Hilliard and Reis (1998) [51] present a quasi analytical solution for standard options using this model. Despite, as Clewlow and Strickland (2000) [16] suggest, this solution appears to present faults because it is not consistent with the attenuation of the jumps in the case of mean reversion.

In summary, the inclusion of stochastic interest rates in the commodity prices models does not have a significant impact in the pricing of commodity futures and options in practice.

Cortazar and Schwartz (2003) [20] propose another three-factor model which extends Gibson and Schwartz (1990) [42] and Schwartz (1997) [82] two-factor model. However, the third factor is not a stochastic interest rate but instead a long-term spot price return in the spirit of Pilipovic (1997) [74] and Schwartz and Smith (2000) [84]. The authors first rewrite the Schwartz (1997) [82] two-factor model in terms of the spot price diffusion and a demeaned convenience yield, y , which is obtained by subtracting the long-term convenience yield, α , from the initial convenience yield, δ , that is:

$$y = \delta - \alpha \quad (2.56)$$

This stochastic third factor follows a O-U stochastic process. The dynamics of this three-factor model is:

$$dS = (\nu - y)Sdt + \sigma_1 Sdz_1 \quad (2.57)$$

$$dy = -\kappa ydt + \sigma_2 dz_2 \quad (2.58)$$

$$d\nu = a(\bar{\nu} - \nu)dt + \sigma_3 dz_3 \quad (2.59)$$

with

$$dz_1 dz_2 = \rho_1 dt, \quad dz_2 dz_3 = \rho_2 dt, \quad dz_1 dz_3 = \rho_3 dt \quad (2.60)$$

where y represents a demeaned convenience yield δ , obtained by subtracting the long-term convenience yield α , ν is the long-term price return. This model also has a closed form solution for futures prices. The authors argue that this model performs better than the existing two-factor models in the literature. However, after analyzing the resulting pricing errors when evaluating futures prices we observe that this model performs well for the medium term maturity futures contracts but worsens for short and long term maturities. Although the log futures pricing formula is linear in the model state variables, the authors forego the Kalman filter techniques used previously in the literature to estimate the parameters. Alternatively, they implement the least-squares method starting with pre-determined values, estimating the state variables for each day of the sample and obtain new parameter values recursively until convergence.

For tractability purposes, all these multi-factor models, with the exception of Gibson and Schwartz (1990) [42], assume that the market price of risk of the model state variables is constant. As pointed out by Schwartz and Smith (2000) [84] and by Cortazar and Schwartz (2003) [20], it would be more appropriate to establish a dependence of the risk premium on the respective state variable. These authors suggest that this relationship could be represented by a linear regression on the respective variable. More specifically, the risk premium would be represented by a linear regression on the variable. However, this extension would imply the additional estimation of two parameters for every risk premium. Consequently, the authors choose to assume that the risk premium is constant.

On the other hand, Gibson and Schwartz (1990) [42] assume that the convenience yield market price of risk is, at the most, a function of the commodity

spot price, the convenience yield and time. In order to investigate whether the assumption of a constant market price of risk would influence their model's performance they calibrate this parameter twice. First the calibration is undertaken by assuming that the convenience yield market price of risk is constant. After that, the calibration of the convenience yield market price of risk is carried out relaxing this assumption. For this purpose, Gibson and Schwartz employed a "roll over" strategy aimed at estimating the convenience yield market price of risk over a shorter time period and using the subsequent time period to test the model. Accordingly, they divided the total time period covered by the data in smaller sub-periods. They observed that the convenience yield market price of risk fluctuates among successive time intervals. They concluded that the results for all maturities futures contracts improves when they take the variability of the convenience yield into account. Nevertheless, we would like to point out that these results may also have been affected by the estimation method used by Gibson and Schwartz, which requires the calculation of proxies for both the spot price and the convenience yield.

In summary, we conclude that the assumption of a constant convenience yield market price of risk may reduce the accuracy of futures and options valuation. On the other hand, the assumption of the convenience yield market price of risk as a function of the variables, as suggested by Schwartz and Smith (2000) [84] and by Cortazar and Schwartz (2003) [20], implies the additional calibration of two parameters. This implies a more complex overall calibration process, which may also affect the accuracy of the estimation. This explains why most of the reduced form models presented in the literature assume a constant market price of risk.

Although all the multi-factor models presented above generate a rich set of dynamics for the commodity term structure and represent powerful tools for derivatives

pricing, they also seem to present some problems. First, the treatment of the convenience yield may appear elusive due to the fact that the convenience yield is a non-observable variable which is derived from the relation between futures and contemporaneous spot prices. Accordingly, it is an endogenous variable to the spot and futures prices dynamics. In addition, these models do not guarantee that the convenience yield is always positive. In other words, there is no guarantee that the cash-and-carry arbitrage-free condition is always satisfied. More specifically, arbitrage-free arguments require that the discounted futures prices net of carrying costs cannot be greater than the discounted contemporaneous spot prices. In the event of a negative convenience yield, the following cash-and-carry arbitrage-free condition is violated⁸:

$$F_{t,T} \leq S_t \exp \{(r + c)(T - t)\} \quad (2.61)$$

where $F_{t,T}$, S_t are as in (2.1), r is the risk free rate and c is the storage cost expressed as a proportion of the spot price.

Secondly, as Pirrong (1998) [75] points out, these models have other misspecification problems due to the fact that they consider that both the spot price and the convenience yield have constant volatility and constant correlation. In other words, they do not allow state-dependent volatilities and the correlation between the two state variables to depend on inventory levels implied by the theory of storage. In particular, it is well established by the theory of storage that commodity spot prices are heteroscedastic and that the spot volatilities depend on the level of inventory and the correlation between spot prices and convenience yields is an increasing function of inventory levels. The existing literature overlook these important properties and therefore generate option pricing errors. This is also pointed out by Routledge, Seppi and Spatt (2000) [78]) and Clewlow and

⁸This condition follows easily from cash-and-carry arbitrage-free arguments.

Stickland (2000) [16]. The disregard of these properties show that there this class of models and the structural models in the literature are largely disjoint.

Heath, Jarrow and Morton (HJM) Approach for Commodity Price Models

The interest rate models models within this class are based on the approach to valuation of interest-rate claimed by Ho and Lee (1986) [52] and Heath *et al* (1992) [50]. These models differ from the standard approach as follows. Instead of making assumptions about the instantaneous interest rate process and its local premium, they specify the volatility of the price process of all pure discount bonds and their initial prices, or alternatively, the initial term structure of forward rates and the way it evolves through time. They derive an interest rate claims model that uses these variables as an input, study the consistency of their model with no arbitrage, and derived the relationship between the inputs of their model to the instantaneous interest rates process and its risk-premium. Reisman (1991) [76] suggests the use of the same approach to model commodity prices following this framework by specifying the futures price processes for all maturities conditional on the initial futures prices term structure under the risk-neutral measure. The corresponding contingent claim model uses as inputs the local volatilities of futures prices of all maturities and their current prices. Within this approach the models are defined by the futures price processes for all maturities, where the spot prices and the convenience yield processes are implied by these inputs. In contrast, in the standard approach the model is defined by the specification of the spot price and the convenience yield processes. Although both approaches are equivalent in the sense that the inputs to each model imply the inputs to the other, the HJM framework presents a great simplification to the standard one. This is due the fact that the instantaneous expected return on all future contracts (which require

no investment) is equal to zero under the risk neutral measure. Therefore, we can evaluate futures prices and commodity contingent claims without the need of explicitly modelling the drift of the spot price stochastic process.

Reisman (1991) [76] is the first to adopt this perspective to evaluate commodity claims and show that the standard spot diffusion approach and the HJM approach are equivalent. More specifically, this approach defines a commodity price model by specifying the local volatilities of futures prices of all maturities and their current prices. The corresponding spot price and convenience yield processes can be derived from any arbitrary specification of these inputs. Cortazar and Schwartz (1992) [18] apply this model to analyze daily copper futures traded at the Commodity Exchange of New York (COMEX) over the period January 1966 - January 1991. They apply Principal Component Analysis to daily copper futures prices to obtain a three-factor model that describes the stochastic movement of futures prices. Additionally, they price publicly traded Copper Interest-Indexed Notes using simulation techniques. The model described in Cortazar and Schwartz (1992) [18] is as follows.

Let $S(t)$ be the spot price of a commodity at time t and $F(t, T)$ the price of a futures contract at time t , written on the same commodity, for delivery at time T . It is also assumed that the futures are traded in a frictionless continuous market, however, trading and storing may be costly. Under the risk neutral measure, the instantaneous return on all futures contracts (which require no investment) is equal to zero (see Cox, Ingersoll and Ross, 1981 [22]).

The risk-adjusted process for the commodity futures price is given by:

$$\frac{dF(t, T)}{F(t, T)} = \sum_{k=1}^K b_k(t, T) dW_k \quad (2.62)$$

or equivalently:

$$F(t, T) = F(0, T) + \int_0^t \sum_{k=1}^K b_k(s, T) F(s, T) dW_k(s) \quad (2.63)$$

where W_1, W_2, \dots, W_K are K independent Brownian motions under the equivalent martingale-measure, and $b_k(t, T)$ are volatility functions of futures prices. From this specification of the futures prices process we can obtain the process for the spot price. Equation (2.63) can be rewritten as:

$$F(t, T) = F(0, T) \exp \left(-\frac{1}{2} \int_0^t \sum_{k=1}^K b_k^2(s, T) ds + \int_0^t \sum_{k=1}^K b_k(s, T) dW_k(s) \right) \quad (2.64)$$

By setting $T = t$, we determine the process for the spot price $S(t)$:

$$S(t) = F(0, t) \exp \left(-\frac{1}{2} \int_0^t \sum_{k=1}^K b_k^2(s, t) ds + \int_0^t \sum_{k=1}^K b_k(s, t) dW_k(t) \right) \quad (2.65)$$

Alternatively, the risk-adjusted stochastic process for the spot price $S(t)$ could have been written as:

$$\frac{dS(t)}{S(t)} = y(t)dt + \sum_{k=1}^K c_k(t) dW_k(t) \quad (2.66)$$

or equivalently as:

$$S(t) = S(0) + \int_0^t y(s) S(s) ds + \int_0^t \sum_{k=1}^K c_k(s) dW_k(s) \quad (2.67)$$

in which $y(t)$ represents the instantaneous cost of carry for investing in the commodity and $c_k(t)$ its volatility parameters. Reisman (1991) [76] and Cortazar and Schwartz (1992) [18] interpret $y(t)$ as the instantaneous risk free return obtained by buying one unit of the commodity spot and selling one futures contract in the next instant of time, or instantaneous cost-of-carry. Following no-arbitrage arguments $y(t)$ is equal to the risk free interest rate, $r(t)$, minus the net convenience

yield, $\delta(t)$ ⁹. This instantaneous cost-of-carry is stochastic. The process followed by $y(t)$ is derived from applying the Ito's lemma to equation (2.65) and compare the corresponding drift and stochastic terms with those of equation (2.66), obtaining:

$$c_k(t) = b_k(t, t) \quad (2.68)$$

$$y(t) = \frac{\partial F(0, t)}{\partial t} - \int_0^t \sum_{k=0}^K b_k(s, t) \left(\frac{\partial b_k(s, t)}{\partial t} \right) ds + \int_0^t \sum_{k=0}^K \left(\frac{\partial b_k(s, t)}{\partial t} \right) dW_k(s) \quad (2.69)$$

Therefore, the specification of the process for the futures prices completely determines the process for the spot prices under the risk neutral measure. Reisman (1991) [76] shows that the problem can also be formulated in terms of forward cost-of-carry instead of futures prices. Specifying both the risk-adjusted process for the forward cost of carry and for the spot price is equivalent to specifying the risk-adjusted process for futures prices. Let $y(t, T)$ be the instantaneous forward cost of carry of investing in the commodity at time T , as perceived at time t . Similarly to the definition of $y(t)$, $y(t, T)$ is the instantaneously riskless forward return obtained by buying one futures contract with maturity T and selling one with maturity $T + dT$. Then,

$$F(t, T) = S(t) \exp \left(\int_t^T y(t, s) ds \right) \quad (2.70)$$

Let's write the stochastic process for $y(t, T)$ as:

$$dy(t, T) = A(t, T)dt + \sum_{k=1}^K B_k(t, T)dW_k(s) \quad (2.71)$$

Reisman shows that the relation between the parameters of equations (2.71) and

⁹Net convenience yield represents the marginal storage cost minus instantaneous convenience yield.

those in (2.66) are given by:

$$A(t, T) = - \sum_{k=1}^K (B_k(t, T)) \left(c_k(t) + \int_t^T B_k(t, s) ds \right) \quad (2.72)$$

$$b_k(t, T) = c_k(t) + \int_t^T B_k(t, s) ds \quad (2.73)$$

$$B_k(t, T) = \frac{\partial b_k(t, T)}{\partial T} \quad (2.74)$$

Thus, given the process for the spot price and for the forward cost of carry, the process for the futures prices can be determined.

Because this model is specified under the risk neutral measure, the valuation of options on futures prices follows easily. First, it is necessary to estimate the model for the movement of futures prices under the equivalent martingale measure and use the risk neutral probabilities to simulate the stochastic paths followed by the futures prices. Second, we can determine all payoffs contingent on these futures prices and discount the expectation of these payoffs at the risk-free rate and obtain an estimation for the value of the contingent claim.

Miltersen and Schwartz (1998) [68] develop a three-factor model using all the information in the initial term structures of both interest rates and commodity future price movements. The model of future price movements is defined by three stochastic differential equations (SDEs) for the continuously compounded forward interest rates, f , the continuously compounded future convenience yield, ϵ and the spot price of the underlying commodity, S . That is:

$$f(s, t) = f(0, s) + \int_0^t \mu_f(u, s) du + \int_0^t \sigma_f(u, s) dW_u \quad (2.75)$$

$$\epsilon(s, t) = \epsilon(0, s) + \int_0^t \mu_\epsilon(u, s) du + \int_0^t \sigma_\epsilon(u, s) dW_u \quad (2.76)$$

$$S_t = S_0 + \int_0^t S_u \mu_s(u) du + \int_0^t S_u \sigma_s(u) dW_u, \quad (2.77)$$

where W is a standard d -dimensional Wiener process. Possible correlations among the three processes comes via the specification of the diffusion terms (the σ_s), since it is the same Wiener process, W , that is used in all three SDEs. The drift terms (the μ_s) and the diffusion terms (the σ_s) are not specified. Naturally, these processes must fulfill certain regularity conditions and the usual arbitrage-free condition that implies that the drift of the spot price is:

$$\mu_s(t) = r_t - \sigma_t \quad (2.78)$$

under the risk-neutral measure. However nothing is mentioned relatively to the negativity values of the convenience yield, which may generate cash-and-carry arbitrage possibilities. Additionally, a general volatility structure may not be suitable to fit the data. Some numerical examples are provided by this paper but no empirical work is implemented to test this model.

Clewlow and Strickland (1999) [15] also use the HJM to price a wide range of energy derivatives by estimating the volatilities and correlations from the market data. They develop a single-factor model which is a particular case of the general multi-factor model in Reisman (1991) [76] and Cortazar and Schwartz (1992) [18]. In particular, the evolution of the forward curve is given by:

$$\frac{dF(t, T)}{F(t, T)} = \sigma_1(t, T)dz \quad (2.79)$$

where

$$\sigma_1(t, T) = \sigma \exp\{-\alpha(T - t)\} \quad (2.80)$$

is the forward price volatilities term structure extracted implied by Schwartz (1997) [82] single-factor model. This is the most simple model consistent with attenuating volatility, which is consistent with the exponential decreasing forward volatilities observed in the market. Moreover, in order to obtain a Markovian spot price process the volatilities of the forward prices must have a negative exponential

form. They present the analytical solutions for standard options, caps, floors, collars and swaptions and demonstrate the application of the trinomial tree, which are constructed to be consistent with the forward curve and volatility structure. Although this paper presents interesting analytical results, it has not been tested empirically.

Manoliu and Tompaidis (2000) [65] also present a model that fits the general HJM framework. Here, the spot price is assumed to be a given function of underlying state variables. These variables follow generalized O-U process under the risk-neutral measure. The futures price is then given by the risk-neutral conditional expectation of the underlying spot at the maturity of the futures price. They account for seasonal patterns in the future curve and also offer a Kalman filter formulation for the general model. Moreover, they apply their model to study the natural gas-market with one and two factors, taking into account seasonal stochastic processes with one and two factors.

Let S_t denote the spot price of the energy commodity at time t . Manoliu and Tompaidis assume that S_t can be decomposed as the product of several components, one of which might be a seasonality factor. More precisely, if:

$$X_t = \ln S_t \quad (2.81)$$

the authors assume that X_t can be expressed as a sum of the m state variables $\xi_t^1, \xi_t^2, \dots, \xi_t^m$, that is:

$$X_t = \sum_{i=1}^m \xi_t^i \quad (2.82)$$

The state variables ξ_t^i are assumed to each follow a stochastic process defined under the risk neutral measure Q by the stochastic differential equation:

$$d\xi_t^i = (\tilde{\alpha}_t^i - k_t^i \xi_t^i)dt + \sum_{j=1}^n \sigma_t^{ij} \widetilde{dW}_t^j \quad (2.83)$$

with k_t^i , σ_t^{ij} and $\tilde{\alpha}_t^i$ deterministic functions of time. For $\sigma_t^{ij} = \sigma^{ij}$ and $k_t^i = k^i$ non-zero constants, the above SDE defines an O-U process, with mean-reverting rate k^i and mean-reverting level $\frac{\tilde{\alpha}_t^i}{k^i}$, while for $k^i = 0$ it reduces to a Brownian motion process with drift $\tilde{\alpha}_t^i$.

This model is consistent with no-arbitrage pricing in the sense that futures contracts and contracts with payoffs based on futures prices are martingales under the risk-neutral measure. However, the (discounted) spot price is not a martingale under the risk neutral measure, which is not consistent with no-arbitrage arguments. To justify this inconsistency, the authors argue that the spot price in their model does not correspond to the price of a tradable asset and it is therefore not observable.

For the model estimation, the authors assume that:

- The market price of risk is constant;
- The number of state variables is $m = n + 1$, with $\xi_t^1, \xi_t^2, \dots, \xi_t^n$ stochastic components and with $q(t) = \xi_t^{n+1}$ the seasonality component, assumed constant and periodic, with period set to be equal to one year.
- The quantities, k_t^i , σ_t^{ij} and $\tilde{\alpha}_t^i$, are constant.
- Time-homogeneous instantaneous volatilities for futures prices.

The authors test both single-factor and two-factor model using natural gas futures daily data with maturities that vary between one and fifteen months over the period September 1997-August 1998. The single-factor model is described by a deterministic seasonality factor and one random factor, which follows an O-U process. The two-factor model assumes the same deterministic seasonality factor plus two stochastic components. One follows a O-U stochastic process and the other follows a Brownian motion. They compare the empirical volatility structure

with the single-factor and the two-factor model implied volatility. In both cases, the model does a poor job. Regarding the correlation structure between price movements on different futures contracts, the single-factor model does not fit the empirical data, while the two-factor model captures the observed correlation structure of futures prices. The fit to the forward curve is better for the two-factor model than the single-factor model. Generally, the two-factor model fits the data better than the single-factor model by comparison of likelihood scores and the standard deviation of prediction errors. Additionally, the two-factor model is able to capture the observed correlation structure of futures prices. However, both models perform poorly on the short term. More specifically, the fit of both models volatilities to the empirical ones is rather poor for the short-term contracts. Moreover both models have high standard deviations for short-term contracts.

Although the HJM approach considerably simplifies the standard model, it also presents limitations. As with the spot price diffusion methodology, there is no restriction in the contango relationship between spot and futures commodity prices allowing for cash-and-carry opportunities. Additionally, none of these studies provide a realistic model that fits the data well. This seems to be due to the general futures volatility these models assume, which do not fit the data. This suggests that further studies should be pursued using better volatility specifications.

2.4 Conclusion

This research develops new models in continuous time for the price of a storable commodity and futures valuation that overcome some of the problems associated with the current literature. In this chapter we have reviewed the main concepts necessary to understand the theory of storage. We also critically reviewed the commodity price models which belong to the reduced form models and to the

structural models. Moreover, we reviewed one partial equilibrium model that does not belong to either of these two categories but is being considered as important in the context of this thesis. In this chapter we discussed the advantages and drawbacks of each of the approaches. Additionally, we pointed out that the two main streams of the literature in commodity price modelling are largely disjoint. In particular, we argued that some of the misspecifications presented by the existing models are mostly due to this disconnection. In particular, existing structural models are developed under rigid discrete time framework that does not take into account the mean-reverting properties of commodity prices. Furthermore, most of the literature within this class does not analyze the properties of the futures prices. Current reduced form models allow cash-and-carry arbitrage possibilities and do not take into account the dependence between the spot price volatility and the inventory levels. Therefore, one of the goals pursued in this thesis is to establish a link between two apparently disjoint streams of the literature in commodity price models - the structural models and the reduced form models. Taking this into account, we describe one structural model and two reduced form models where each of these represents an advance of their type.

First we develop a stochastic structural model formulated using the stochastic dynamic approach in continuous time. This stochastic equilibrium model considers that the source of uncertainty comes from the supply¹⁰ and expands the discrete time structural models in the literature reviewed in four different aspects. First, we include mean reversion in this model through the supply specification, that is, the supply rate evolves as a O-U stochastic process. The inclusion of mean reversion in a structural model is a significant contribution to the literature for two reasons. One is that this property is a key property in the empirical behaviour of

¹⁰The supply can be interpreted as the difference between supply and the stochastic part of the demand. This interpretation is appropriate since most of the commodities have stochastic demand.

commodity prices, which is largely acknowledged by the existing reduced form models but not taken into account by the structural models. In fact, none of the existing structural models has included the mean reversion property of commodity prices. The other is that we establish a link between structural models and reduced form models. Second, we formulate this model for both competitive and monopolistic storage environments. By doing so we provide a valuable comparison between competitive and monopolistic storage policies and how these differences are reflected in the price dynamics. This comparison is not illustrated in the current storage models, which only consider a competitive storage economy. Since the energy markets have evolved from a monopolistic to a competitive environment in recent years, we stress the importance to analyze both storage economies in order to understand the implications of the market evolution in terms of the price dynamics. Third, this model is developed under a very flexible framework which allows for different extensions of the model to be adapted to different commodities. In particular, we explain how seasonality can be included in the model without adding extra complexity to the solution method. Finally, we provide a numerical analysis of the equilibrium forward curves implied by this structural model. This is also an important contribution to the literature. More specifically, with the exception of Routledge, Seppi and Spatt (2000) [78], the existing literature in structural models for commodity prices limits the scope of analysis to the study of the spot price properties as a function of the state variables. Hence these studies do not analyze the forward curves or any other derivatives implied by the equilibrium models. Routledge, Seppi and Spatt present a study of equilibrium forward curves conditional on the initial inventory demand levels¹¹. Nevertheless, their analysis is limited to the case where the demand can only take two possible states

¹¹As mentioned before, the source of uncertainty in their model comes from the stochastic demand.

- high and low - and it is very difficult to generalize to a more realistic number of demand states or to a more general Markov process. On the other hand, we present a numerical analysis of the forward curve that is broad and can easily be generalized to any combination of initial values of the supply and inventory.

As mentioned before, one of the misspecifications associated with the existing reduced form models is associated with the definition of the convenience yield. These models do not guarantee that the convenience yield is always positive. In other words, there is no guarantee that the cash-and-carry arbitrage-free condition is always satisfied. More specifically, arbitrage-free arguments require that the discounted futures prices net of carrying costs cannot be greater than the discounted contemporaneous spot prices. In the event of a negative convenience yield the cash-and-carry arbitrage-free condition may be violated. The first reduced form model presented in this thesis essentially focuses on replicating the mean reversion characteristic of commodity spot prices and on ensuring absence of arbitrage by restricting the contango relationship between spot and forward prices. We report the statistical properties implied by this model and analyze the corresponding forward curve. We also perform a comparative analysis with the single-factor model presented by Schwartz (1997) [82].

Another misspecification associated with the current reduced form models is the assumption of constant spot price and convenience yield volatilities, which does not reflect the empirical characteristics of storable commodity prices. The theory of storage and the structural models in the literature show that the commodity price volatility is time varying. In particular, the prices of storable commodities should be high volatile when demand and supply conditions are tight (due to high demand and/or low inventories) but exhibit little volatility when supply is abundant (due to low demand and/or high inventories). In other words, commodity spot prices

are heteroscedastic and volatilities are negatively related with the inventory level. The second reduced form model developed in this thesis aims to simultaneously follow cash-and-carry arbitrage free conditions and to take the properties of the commodity price volatilities into account. This is a two-factor model where spot prices and instantaneous convenience yield follow a joint stochastic process with constant correlation in the spirit of Gibson and Schwartz (1990) [42] and Schwartz (1997) [82] and the volatility in both stochastic processes is proportional to the square root of the convenience yield level. Since the level of the convenience is negatively related to the inventory level, this definition of volatility ensures that the spot price and convenience yield volatilities are negatively related with the inventory level, and the spot price is heteroscedastic. Additionally, we also ensure that the convenience yield never becomes negative since we describe it as a Cox, Ingersoll Ross (CIR) (1985) [23] stochastic process. This model is empirically tested using oil futures data over the period 17th of March 1999 - 24th December 2003 and compared with the Schwartz (1997) [82] two-factor model.

Chapter 3

A Structural Model for the Price Dynamics: Competitive and Monopolistic Markets

3.1 Introduction

In this chapter, we present a new stochastic structural model in continuous time for the price of a storable commodity. In this model we assume that the uncertainty arises from the supply, whereby the supply rate evolves as a mean reverting stochastic process of the Ornstein-Uhlenbeck (O-U) type. This model is developed under a general framework which provides two distinct forms as separate cases to represent the alternative economic scenarios of perfect competition and monopolistic storage.

The equilibrium structural models are derived explicitly from economic principles and aim to replicate the equilibrium price for storable commodities under the rational expectations hypothesis. Most of the existing papers focus on establishing

an equilibrium price model for agricultural commodities where the supply is determined by speculative storage and random behavior of harvests. The price is the solution to functional equations relating supply, demand and storage. Due to its complexity, this solution is derived by numerical approximations. The multi-period competitive storage equilibrium is built up and motivated from the perspective of the behavior of individual price taking firms. The non-negative constraint leads to non-linearity in the equilibrium models and is carried into non-linearity of the commodity market price. This approach is standard and is described in Williams and Wright (1991) [89] and is also adopted by Deaton and Laroque (1992, 1996) [24, 25], Chambers and Bailey (1996) [12], Routledge, Seppi and Spatt (2000) [78]. An alternative way of deriving the competitive storage equilibrium prices is from a social planner perspective as to maximize the consumer surplus. The dynamic programming approach to solve rational expectations behavior is first introduced by Gustafson (1958a, b) [43, 44]. Samuelson (1971) [80] associates the competitive storage behaviour to the storage problem. Lucas and Prescott (1971) [77] also present a rational expectations equilibrium price model for investment, output and prices. The equilibrium evolves as *if* to maximize the expected present value of social welfare in the form of "consumer surplus". Both the approaches described are equivalent as shown by Williams and Wright (1991) [89].

The reduced form class of models dominates the current literature and practice on energy derivatives. Leading models include Gibson and Schwartz (1990) [42], Schwartz (1997) [82], Miltersen and Schwartz (1998) [68]. These models consider that the spot price and the convenience yield follow a joint stochastic process with constant correlation. The main focus of these models is to replicate the mean reversion in commodity spot prices. Nevertheless, the use of current reduced form models in the literature to price energy contingent claims has not

been effective. In particular, the convenience yield process seems to be misspecified since its specification ignores some crucial properties of commodity prices behaviour such as the dependency of prices variability on inventory levels (see Pirrong, 1998 [75] and Clewlow and Strickland, 2000 [16]). These misspecifications call for a better understanding of the supply, demand and storage roles on the dynamics of energy prices. Accordingly, the development of new structural models that take the key properties of the energy commodity markets into account is a fundamental tool that help us to understand the interplay between the microeconomic factors that drive the market and the price dynamics. In this context, it is very surprising that current research on structural models for commodity prices does not follow the recent energy markets developments. In particular, none of the existing equilibrium models takes the mean reverting properties of commodity prices into account, which is much stronger in energy markets than in agricultural products. Moreover, none of the existing structural models are tested empirically using energy price data.

Fackler and Livingston (2003) [36] present a partial equilibrium storage model that takes into account both the mean reversion and seasonality properties of commodity prices. This model is formulated as a continuous time stochastic dynamic programming problem. The state variables are the exogenous spot price and the endogenous storage level, whereby the rate of sales is the decision variable. The spot price follows a O-U stochastic process modified to allow seasonal variation in the mean and variance. Although this model does include key properties of commodity prices, such as mean reversion and seasonality, some of its assumptions are unrealistic. First this model assumes that the decision variable does not affect the commodity price. Second, the stockholder, which is also the producer is not allowed to buy the commodity from any other producer or stockholder. Finally, it

does not limit storage capacity. As a result, the model produces a solution of the "bang-bang" type, whereby it is optimal either to keep all its initial stock or to sell everything at an infinite rate. Although this model presents an elegant formulation, it does not represent realistically the dynamic interplay between demand, supply and storage.

The model presented in this chapter is an equilibrium model, which is formulated using the stochastic dynamic approach in continuous time. This model considers two state variables. One is the supply rate, which is an exogenous stochastic variable. The other is the inventory, which is an endogenous variable, whereby the rate of storage is the decision variable in this model. This model expands the current literature on structural models for commodity prices by taking into account the mean reverting characteristics of commodity prices. One of the most innovative features of this model is that it establishes the link between the two major categories of the literature - the discrete time structural commodity price models and the continuous time reduced form models. Specifically, on one hand we draw on a structural model formulation in the fashion of those originally developed to study agricultural commodity prices. On the other hand we simultaneously include the mean reversion in commodity prices, which is the central price characteristic of reduced form models. More specifically, we develop a model based on the microeconomics of supply, demand and storage similar to the structural models and add three important features. First, we consider a continuous time framework whereas the traditional models consider a discrete time framework. Second, we include the mean reverting characteristic of commodity spot prices in the dynamics of our model similar to those proposed in the reduced form models. Particularly, the mean reversion is introduced into the model by considering the exogenous supply as a mean-reverting O-U stochastic process. This mean-reverting process can be

interpreted as the net supply, that is, the difference between the exogenous supply and the stochastic demand in the market. This interpretation is appropriate since uncertainty arises from the demand side in many commodity markets, such as energy. Although we do not consider seasonality in this model, we explain how to extend it to include this property without adding complexity to the numerical computation of the solution. Third, we formulate and analyze separately the model for both competitive and monopolistic storage economies. This comparison is not illustrated in the current structural models literature, which only consider a competitive storage economy.

This model has two key similarities with Fackler and Livingston (2003) [36] storage model. First, both models focus on the mean reversion characteristic of commodity prices. Second, both models apply a continuous time dynamic programming formulation, whereby the decision variable is the storage rate. However, there are several differences between the two models. The key difference is that our model is an equilibrium model whereby the supply and demand functions are explicitly specified. Accordingly, the storage policy affects the market price and the interplay between storage and price is the main focus of our analysis. In contrast, Fackler and Livingston present a partial equilibrium model and consider that the price is not affected by the storage policy, which is unrealistic. Moreover, Fackler and Livingston assume that the producer is also the stockholder that cannot purchase from other stockholders or producers. In our model, producers and stockholders are independent entities and the stockholder is allowed both to buy and sell. Finally, Fackler and Livingston do not consider limitations in the storage capacity while we consider a fixed storage capacity and analyze the outcome of this restriction.

Within the pure competitive market context, we suppose that a single ho-

homogeneous commodity is continuously traded market where storage is purely speculative over a finite-time horizon. This formulation is then extended to a market structure where storage is monopolistic but production and producer sales remain exogenous. We use continuous time stochastic dynamic programming to obtain the optimal storage policy, which determines the price dynamics. Although we keep the model formulation general, we consider a linear inverse demand function to provide numerical examples for simplicity. The analysis of this model mainly focuses on three issues: (i) the dependence of the storage on both the inventory level and the supply rate, (ii) how the storage policy affects changes the evolution of the commodity natural price¹, (iii) how different levels of inventory affect the commodity price variability and (iv) the differences between the competitive and the monopolistic storage policies in terms of (i), (ii) and (iii).

The remaining of this chapter is structured as follows. Section 3.2 formulates the model and describes the solution method. We present the model under a general framework and later unfold it into two distinct market scenarios: competitive and monopolistic market. Section 3.3 describes the numerical implementation. Section 3.4 provides numerical examples for both markets contexts. Section 3.5 explains how to extend the current model to accommodate seasonality in the supply. Section 3.6 concludes.

3.2 Storage Equilibrium

Our analysis builds on and extends the discrete time framework formulated in Williams and Wright (1991) [89]. Williams and Wright develop a basic discrete time model for commodities in a pure competitive market using a discrete time dynamic programming approach. We use the basic storage formulation of Williams

¹By natural price we mean the commodity price evolution in the absence of storage.

and Wright as a starting point and introduce three main features their framework and to the current structural models for the price of a storable commodity. First, we develop the model in continuous time instead of the discrete time setting used in the traditional literature. Second, we introduce mean-reverting properties in the price dynamics by modelling the exogenous supply rate as a mean reverting stochastic process of the O-U type. Finally, we extend the model to the separate case of a monopolistic storage economy and compare it with the competitive setting.

We consider the existence of two state variables. One is the exogenous supply rate and the other is the endogenous inventory level. The stochastic supply rate can be interpreted as the difference between the exogenous supply and the stochastic part of the demand side. This interpretation is appropriate since demand is stochastic for many commodities. In the competitive storage economy, the storage decisions are made from a social planner perspective *as if* to maximize the expected present value of social welfare in the form of "consumer surplus". The planner's problem in the current period, t , is to select the current storage that will maximize the discounted stream of expected future surplus. The decision variable is the rate of storage, that is, the rate at which the commodity is bought or sold by the stockholder², which can be either positive or negative. For the monopolistic storage context, the competitive formulation is modified to consider that the decisions are made by the monopolistic stockholder, which maximizes the discounted stream of expected future cash-flows generated by his storage facility. For each market, the optimal storage policy is defined by specifying the rate of storage for each possible state of the world at each moment in the future.

²Stockholder refers to the aggregate storage in the competitive market.

3.2.1 Model Formulation

Both models are developed using the same basic framework. In one case we assume that the market (including storage) is perfectly competitive; in the other we assume that storage (only) is monopolistic. In the competitive equilibrium, we assume that the number of firms in the storage industry is sufficiently large for each to be a price taker. The storage decisions are made by a single identity, the "invisible hand". Under monopolistic storage, consumers can deal directly with producers through the market but neither group can store on its own. Only one firm has the right or the technology to store the commodity. A monopolistic firm is not the only source of the commodity for consumers since the producers also continuously supply the market. Hence, the monopolist does not extract its extra profits by holding the commodity off the market to keep the price high. Likewise, the firm competes with consumers for any quantity it purchases on the market. The model is developed under the risk neutral measure whereby all the economic agents are risk neutral.

We introduce the model under a general formulation and later unfold it into the two distinct market scenarios.

The general assumptions of the model are as follows:

- A single homogeneous commodity is produced and traded in continuous time, over a finite-time horizon T ;
- Storage is purely speculative, whereby inventory decisions are driven by the single motive of trading profit;
- The supply has zero elasticity;
- The marginal storage cost, k , is constant; the storage cost is $k \times s$ per unit of time, where s is the current storage level;

- The one-period risk-free interest rate, $r \geq 0$ is constant.

We consider two state variables: the exogenous supply rate and the inventory level. The exogenous supply rate, z_t , is given by:

$$dz_t = \alpha(\bar{z} - z_t)dt + \sigma dB_t, \quad t \geq 0 \quad (3.1)$$

where:

- α is the speed of mean reversion;
- \bar{z} is the long-run mean, that is, the level to which z reverts as t goes to infinity;
- σ is the (constant) volatility;
- B_t is a standard Wiener process.

The aggregate storage level, s , is a fully controllable endogenous state variable and satisfies:

$$ds = u(s, z, t)dt, \quad s \geq 0 \quad (3.2)$$

where u represents the rate of storage and is the decision variable in our problem. At each time t , the rate at which the commodity is stored depends on the amount already in storage, s , and on the exogenous supply, z .

Note that the decision $u(\cdot)$ is a function in $[0, T]$, which we call the *inventory management plan*. If the inventory capacity is $b > 0$, then the inventory level $s(t)$ must satisfy the constraint:

$$0 \leq s(t) \leq b \quad (3.3)$$

since negative storage is not allowed. On the other hand, if $z(t)$ is the supply rate at time t , then $u(\cdot)$ must not exceed this rate, that is:

$$u(t) \leq z(t). \quad (3.4)$$

Constraints (3.3) and (3.4) imply that the optimal storage rate, u^* , belongs to $[u_{\min}, u_{\max}]$ whereby the values u_{\min} and u_{\max} are such that these two constraints are satisfied. Any inventory management plan that satisfies these conditions is called an *admissible plan*. The total rate of consumption in the market, q , establishes the relationship between the state variables defined above and satisfies the equilibrium condition:

$$q = z - u \quad (3.5)$$

Moreover, the market price (or inverse demand function) is given by $p(q)$, where $\frac{\partial p}{\partial q} < 0$.

We consider a finite-time horizon T , at which there is no carryover and we work backwards in time. The following function is then to be maximized:

$$J(s_t, z_t, t; u(\cdot)) = \{E_t \int_t^T e^{-r(l-t)} L(s_l, z_l, u_l, l) dl + \Psi(s_T, z_T) | s = S, z = Z\}. \quad (3.6)$$

The optimization is over all the admissible plans where $L(s_t, z_t, u_t, t)$ is the *instantaneous profit rate* and $\Psi(s_T, z_T)$ is the *salvage value* of having s_T and z_T as states at final time T . Without loss of generality, we consider $\Psi(s_T, z_T) = 0$. The crucial difference between the pure competitive and the monopolistic storage problem formulations consists in the definition of L , which we will describe later.

To find a solution to the problem we use the dynamic programming approach. Accordingly, we need to maximize a value function, $J(\cdot)$, in order to obtain the optimal set of carryover decisions through time. In other words, we apply the Bellman's principle of optimality (Bellman, 1957 [4]). This principle states that at any point of an optimal trajectory, the remaining trajectory is optimal for the corresponding problem initiated at that point. We then obtain the dynamic programming equation of the form (see derivation in appendix A):

$$-\frac{\partial V(s, z, t)}{\partial t} - H(s, z, V_s, V_z, V_{zz}) = 0 \quad (3.7)$$

where:

$$\begin{aligned} H(s, z, V_s, V_z, V_{zz}) = & \sup_{u \in [u_{min}, u_{max}]} \{L(s, z, u, t) + uV_s(s, z, t) + \\ & \alpha(\bar{z} - z)V_z(s, z, t) + \frac{1}{2}\sigma^2 V_{zz}(s, z, t) - \\ & rV(s, z, t)\} \end{aligned}$$

for the value function $V(s, z, t)$ with the boundary condition $V(s, z, T) = 0$. This yields the optimal u^* . Note that u^* needs to be formulated in such a way that the storage constraints are not violated, that is $u_{min} \leq u^* \leq u_{max}$. If u^{unc} represents unconstrained the maximum of the above dynamic programming equation, then:

$$u^* = u_{max}, \quad \text{if} \quad u_{max} \leq u^{unc}; \quad (3.8)$$

$$u^* = u^{unc}, \quad \text{if} \quad u_{min} \leq u^{unc} \leq u_{max}; \quad (3.9)$$

$$u^* = u_{min}, \quad \text{if} \quad u^{unc} \leq u_{min}; \quad (3.10)$$

Finally, the current price is given by:

$$p(q) = p(z - u^*) \quad (3.11)$$

In what follows, we separate the formulation into the competitive and the monopolistic scenarios. The distinction between these two formulations is imposed by the definition of the instantaneous profit rate, $L(s_t, z_t, u_t, t)$, which differs among these two contexts as mentioned above.

Competitive Market

As mentioned above, the competitive equilibrium evolves as if the maximization is made from a social planner perspective. This perspective is also adopted by Samuelson (1971) [80], Lucas and Prescott (1971) [77] and Williams and Wright (1991) [89]. The social planner, in the current period t , aims to select the current rate of storage that will maximize the discounted stream of expected future consumer surplus. Let $p(q)$ represent the inverse demand function and also let:

$$f(x) = \int_0^x p(q) dq, \quad \text{for } x \geq 0, \quad (3.12)$$

Accordingly:

$$L(s_t, z_t, u_t, t) = f(z_t - u_t) - k s_t \quad (3.13)$$

where k is the constant marginal storage cost per period.

The functional to be maximized is then:

$$J(s_t, z_t, t; u(\cdot)) = E_t^z \left\{ \int_t^T e^{-r(l-t)} (f(z_l - u_l) - k s_l) dl + \Psi(s_T, z_T) | s = S, z = Z \right\} \quad (3.14)$$

and we obtain the following dynamic programming equation:

$$-\frac{\partial}{\partial t}V(s, z, t) - H(s, z, V_s, V_z, V_{zz}) = 0 \quad (3.15)$$

where

$$H(s, z, V_s, V_z, V_{zz}) = \sup_{u \in [u_{min}, u_{max}]} \{f(z - u) - ks + uV_s(s, z, t) + \alpha(\bar{z} - z)V_z(s, z, t) + \frac{1}{2}\sigma^2 V_{zz}(s, z, t) - rV(s, z, t)\}. \quad (3.16)$$

By differentiating the right hand side of the above equation with respect to u , we obtain the first order condition that allow us to find the maximum:

$$-f'(z - u) + V_s(s, z, t) = 0 \quad (3.17)$$

A necessary, but not sufficient, condition in order to have a maximum is³:

$$f''(z - u) \leq 0 \quad (3.18)$$

where f' and f'' represent the first and the second order derivative of the function $f(\cdot)$ defined by equation (3.12).

Let $D(\cdot) = p^{-1}(x)$, $x \geq 0$ represent the demand function. If we initially ignore the fact that u^* needs to satisfy the storage constraint, the (unconstrained) maximum, u^{unc} , is given by:

$$u^{unc} = z - D(V_s^*) \quad (3.19)$$

Then, by taking into account the constraints given by (3.3) and (3.4) we obtain the optimal control value, u^* .

³A rigorous mathematical verification of existence and uniqueness of the solution requires additional technical work and is beyond the scope of this thesis.

Monopolistic Market

We now specify the problem for the case of a monopolistic stockholder. In this case, the monopolistic storage manager in the current period, t , aims to select the current rate of storage that will maximize the discounted stream of expected future cash flows generated by the management of his storage facility. The control variable, u , represents the rate of storage, that is, the absolute change in inventory level over an infinitesimally small interval of time; hence $-u$ is the amount he sells over each period to generate profits. The instantaneous rate of profit is given by the proceeds from sales minus the cost incurred on the currently held stocks, that is:

$$L(s_t, z_t, u_t, t) = -u_t p(z_t - u_t) - k s_t \quad (3.20)$$

The functional to be maximized is:

$$J(s_t, z_t, t; u(\cdot)) = \left\{ E_t \int_t^T e^{-r(l-t)} (-u_l p(z_l - u_l) - k s_l) dl + \Psi(s_T, z_T) \mid s = S, z = Z \right\} \quad (3.21)$$

where $\Psi(s_T, z_T)$ represents the salvage value as before. The resulting dynamic programming equation is:

$$-\frac{\partial}{\partial t} V(s, z, t) - H(s, z, V_s, V_z, V_{zz}) = 0 \quad (3.22)$$

where

$$H(s, z, V_s, V_z, V_{zz}) = \sup_{u \in [u_{min}, u_{max}]} \left\{ -u p(z - u) - k s + u V_s(s, z, t) + \alpha(\bar{z} - z) V_z(s, z, t) + \frac{1}{2} \sigma^2 V_{zz}(s, z, t) - r V(s, z, t) \right\} \quad (3.23)$$

We obtain the unconstrained control, u^{unc} by solving the first order condition of the right hand side of the above equation given by:

$$-p(z - u) + up'(z - u) + V_s = 0 \quad (3.24)$$

where p' denotes the first order derivative of the function $p(q)$. The necessary (but no sufficient) second condition to obtain maximum is⁴:

$$2p'(z - u) - up''(z - u) \leq 0 \quad (3.25)$$

where p'' denotes the second order derivative of the function $p(q)$.

Depending on the inverse demand function considered, there might not exist an explicit expression for u^{unc} , therefore equation (3.24) might need to be solved numerically. We then obtain the optimal control u^* taking into account the admissibility constraints.

Boundary Conditions

Since the Bellman equation is a backward equation, the temporal side condition is a final condition, rather than an initial condition. Supposing that no salvage value remains at the final time ⁵:

$$V^*(s, z, T) = 0 \quad (3.26)$$

⁴As before, the proof of existence and uniqueness of the solution is beyond the scope of this study.

⁵Note that this assumption is for simplicity and not a restriction to the method.

In this problem there are no explicit boundary specifications, so the boundary values must be obtained by integrating the Bellman equations along the boundaries (see Hanson, 1996 [46]). The non-existence of explicit boundary conditions implies that there are no exterior circumstances in the nature of the problem that would force it to have specific solutions at the boundaries. Therefore, the boundary version of the Bellman equation will be the same as the interior version of the Bellman equation represented by equations (3.15) and (3.22) with the boundary values applied.

3.2.2 Linear Inverse Demand Function

Although the general formulation of our model allows for different definitions of the inverse demand function $p(q_t)$, we use a linear inverse demand function in the numerical implementation of the model:

$$p(q_t) = a - bq_t, \quad a, b > 0. \quad (3.27)$$

In this case the integral in equation (3.12) for $x = z_t - u_t$ becomes:

$$f(z_t - u_t) = \int_0^{z_t - u_t} p(q_t) dq_t = \int_0^{z_t - u_t} a - bq_t dq_t \quad (3.28)$$

$$= a(z_t - u_t) - b \frac{(z_t - u_t)^2}{2} \quad (3.29)$$

This integral exists and is finite for all values $u_t \leq z_t$. The instantaneous rate of profit $L(s_t, z_t, u_t, t)$ defined in equation (3.13) becomes:

$$L(s_t, z_t, u_t, t) = a(z_t - u_t) - b \frac{(z_t - u_t)^2}{2} - ks_t \quad (3.30)$$

where k is the constant marginal cost of storage.

Now we need to check if the solution to the optimal storage rate, u^* is well defined within each of the market contexts. The inverse demand function is given by $D(x) = \frac{a-x}{b}$. Accordingly, equation (3.19) becomes:

$$u^{unc} = \frac{V_s^* + bz - a}{b} \quad (3.31)$$

which exists and is finite for any V_s^* and z .

For the monopolistic market, the optimal storage is the solution to equation (3.24), which now becomes:

$$u^{unc} = \frac{V_s^* + bz - a}{2b} \quad (3.32)$$

which exists and is finite for any V_s^* and z . The optimal storage rate, u^* is obtained by taking into account the state constraints defined by equations (3.3) and (3.4).

3.3 Numerical Implementation

The solution to the general stochastic dynamic programming problem defined by equation (3.6) is obtained by solving the PDE (3.7) subject to the final condition $V(s, z, T) = 0$. The competitive market problem is defined by equation (3.14) and the corresponding solution is found by solving the PDE described by equation (3.15). Similarly, the monopolistic problem is defined by equation (3.21) and the resulting PDE is described by equation (3.22). These PDEs do not have an analytical solution and therefore it is necessary to apply numerical methods to solve them. The optimal feedback control $u^*(s, z, t)$ is computed as the argument of the maximum in the functional control term $L(s, z, u, t)$.

Despite having a nonlinear partial differential equation (PDE) for both problems, the application of an explicit standard method (e.g. Morton and Mayers, 1994 [69]) to obtain a numerical solution is as good as alternative methods which are more complex and imply a greater computational effort. For comparison we implemented both the explicit standard method and the hybrid extrapolated predictor-corrector Crank-Nicholson method, modified to account for the non-linearities and discontinuities in the PDEs (Hanson, 1996 [46] and Hanson and Ryan, 1998 [47]) were implemented. Both achieved very similar results. Therefore, we adopt the standard explicit numerical procedure to solve PDEs.

3.3.1 Space-time Discretization and the Explicit Solution

In this section we describe the standard explicit method used to solve the PDEs (3.15) and (3.22). The dependent variable $V(s, z, t)$ is represented on a uniform grid in space and time.

The two independent variables for both PDEs are storage, s , and exogenous supply, z . Both spaces have lower and upper limits, which are denoted by s_{Min} , s_{Max} and z_{Min} , z_{Max} , respectively. We divide the storage space domain, $[s_{Min}, s_{Max}]$, into $N_s - 1$ equally spaced intervals of size Δs . Analogously, the space interval representing exogenous supply, $[z_{Min}, z_{Max}]$, is divided into $N_z - 1$ equally spaced intervals of size Δz . The time interval $[0, T]$ is divided into $N_t - 1$ equally spaced intervals of size Δt , that is:

- $s_i = (i - 1) \Delta s$, $i = 1, \dots, N_s$ for the inventory level, where $\Delta s = \frac{s_{Max}}{N_s - 1}$;
- $z_j = (j - 1) \Delta z$, $j = 1, \dots, N_z$ for the exogenous rate of supply, where $\Delta z = \frac{z_{Max}}{N_z - 1}$;
- $t_k = T - (k - 1) \Delta t$, $k = 1, \dots, N_t$ for time, where, $\Delta t = \frac{T}{N_t - 1}$;

$V(i, j, k)$ denotes the numerical approximation to the dependent variable $V(s_i, z_j, t_k)$. If the numerical solution exactly agrees with the true solution, then $V(i, j, k) = V(s_i, z_j, t_k)$. A numerical solution should then satisfy $\lim_{\Delta s, \Delta z, \Delta t \rightarrow 0} |V(i, j, k) - V(s_i, z_j, t_k)| = 0$.

The basic methodology of the finite-difference schemes is to approximate the derivatives appearing in the PDE with combinations (differences) of the values of the grid. A variety of different approximations are possible. For the spatial derivatives we use second-order central differences such that:

$$V_s(s, z, t) \simeq DVS(i, j, k) = \frac{1}{2\Delta s} (V_{i+1, j, k} - V_{i-1, j, k}),$$

$$V_z(s, z, t) \simeq DVZ(i, j, k) = \frac{1}{2\Delta z} (V_{i, j+1, k} - V_{i, j-1, k}).$$

The second-order derivative term $V_{zz}(s, z, t)$ is calculated using the second-order central difference:

$$V_{zz}(s, z, t) \simeq DDVZ(i, j, k) = \frac{1}{(\Delta z)^2} (V_{i, j+1, k} - 2V_{i, j, k} + V_{i, j-1, k}).$$

The backward time derivative $V_t(s, z, t)$ is approximated by:

$$V_t(s, z, t) \simeq DVT(i, j, k) = -\frac{1}{\Delta t} (V_{i, j, k+1} - V_{i, j, k}).$$

According to equation (3.19), the unconstrained control, u^{unc} , for the competitive market is given by:

$$u^{unc}(i, j, k) = (j - 1)\Delta z - D(DVS(i, j, k))$$

where $D(\cdot)$ is the inverse demand function, which is defined in advance. According to equation (3.24), the unconstrained control for the monopolistic storage economy, u^{unc} , is the solution to the equation:

$$-p((j-1)\Delta z - u^{unc}(i, j, k)) + u^{unc}(i, j, k)p'((j-1)\Delta z - u^{unc}(i, j, k)) + DVS(i, j, k) = 0$$

where p' represents the first derivative of the inverse demand function and is defined analytically beforehand. In both cases, the optimal control variable, $u^*(i, j, k)$ is:

$$u^*(i, j, k) = \min(u_{max}, \max(u_{min}, u^{unc}(i, j, k)));$$

The optimal feedback control is then included as an argument in the maximum of the control term in the PDEs given by equations (3.16) and (3.23) for the competitive and the monopolistic market problems, respectively.

Applying an explicit method of solution and moving backwards in time, the discrete extrapolated forward approximation corresponding to the Bellman equation is given by:

$$\begin{aligned} V^*(i, j, k+1) = & V^*(i, j, k) + \Delta t [L^*(i, j, k) + u^*(i, j, k)DVS(i, j, k) \\ & \alpha(\bar{z} - (j-1)\Delta z)DVZ(i, j, k) + \\ & \frac{1}{2}\sigma^2 DDVZ(i, j, k) - rV^*(i, j, k)] \end{aligned}$$

where $L^*(i, j, k)$ represents instantaneous rate of profit for the optimal u^* . This quantity is represented by equations (3.13) and (3.20) for the competitive and for the monopolistic storage economies, respectively.

In summary, starting with the final time solution $V(s, z, T) = 0$ and by moving backwards using this explicit formulation at each time step, we successively obtain the optimal storage policy represented by the optimal storage rate u^* and the subsequent optimal value function, V^* , which, itself, is the solution to the PDE represented by equation (3.15) for the competitive problem and by equation (3.23) for the monopolistic.

Neither problem has explicit boundary conditions (except at the final time T), so the boundary values must be obtained by integrating the Bellman equations along the boundaries (Naimipour and Hanson, 1992 [45]). Consequently, approximations used to calculate the spatial derivatives at the right spatial boundaries are as follows:

- For the spatial first-order derivatives we use the following differences:

$$V_s(s, z, t) \simeq DVS(i, j, k) = \frac{1}{\Delta s} (V_{i, j, k} - V_{i-1, j, k}),$$

$$V_z(s, z, t) \simeq DVZ(i, j, k) = \frac{1}{\Delta z} (V_{i, j, k} - V_{i, j-1, k}).$$

- The second-order derivative term $V_{zz}(s, z, t)$ is calculated using the following approximation:

$$V_{zz}(s, z, t) \simeq DDVZ(i, j, k) = \frac{1}{(\Delta z)^2} (V_{i, j, k} - 2V_{i, j-1, k} + V_{i, j-2, k}).$$

This formulation allows the integration of the Bellman equation along the boundaries without using values for the solution V beyond the spatial domain of the state variables.

3.4 Results

The results reported below consider the optimization problem formulated in equation (3.6). We compute and present separately the results for each alternative economic scenarios of perfect competition and monopolistic storage. We consider a very large final time, that is when $T \rightarrow \infty$ in order to obtain the steady state equilibrium independent of time⁶. In other words, we consider T sufficiently large so that the influence of anticipation that storage will stop in period T becomes

⁶We have also considered the possibility of developing this model for the steady state equilibrium by considering an infinite time horizon. However, the solution would be impossible to obtain without the knowledge of the boundary conditions for s and z .

negligible and the decision rule becomes time independent. We consider $T = 50$ years, which gives a steady state equilibrium for the parameters considered below. As previously described, we perform the analysis for a linear price function: $p(q) = p(z - u) = a - b(z - u)$, $a, b > 0$. Table 1 reports the parameter values used in the numerical implementation and Table 2 specifies the range for the annual supply rate⁷ and the inventory capacity considered. When storage is not available, the price is a function of the exogenous supply only, that is, $p(q) = p(z) = a - bz = 100 - 8z$, $z \in [0.0, 9.0]$. Accordingly, in the long run, the price follows a normal distribution with mean $\mu_p = 64$ and $\sigma_p = 4.6$. Numerous parameter combinations were implemented and analyzed beforehand to ensure that the results reported below are representative for the qualitative model properties.

a	b	α	σ	\bar{z}	k	r
100	8	6.0	2.0	4.5	5.0	0.05

Table 3.1: Value of the Parameters used to obtain the numerical solutions of both competitive and monopolistic markets.

z_{Min}	z_{Max}	s_{Max}
0.0	9.0	0.9

Table 3.2: z_{Min} and z_{Max} represent the lower and upper values of the grid for the exogenous supply rate, z . s_{Max} represents the storage capacity.

Next, we present the results for the optimal storage policy and the resulting price within each of the competitive and the monopolistic market contexts. We also represent graphically the results of the price variability as a function of the

⁷In the long run, z has a gaussian stationary distribution with mean $\mu_z = \bar{z} = 4.5$ and standard deviation $\sigma_z = \frac{\sigma}{\sqrt{2\alpha}} = 0.58$. This range cover the steady state distribution of z .

supply rate at different fixed levels of storage⁸. In the absence of storage, the price volatility is given by $b\sigma = 16$. In the presence of storage, the variability is calculated as σp_z as derived in appendix B, where p_z is the first order partial derivative of the price in order to z . This partial derivative is calculated numerically according to the approximation described before. This illustrates the effect that the existence of storage in the economy has on the variability of the commodity prices. We will also emphasize the differences between the competitive and the monopolistic markets.

3.4.1 Competitive Case

Figure 3.1 shows the optimal storage rate, u^* , as a function of both the storage level and the supply rate. Figure 3.2 illustrates the super-position of the price in the absence of storage and the price in the presence of storage in this economy as a function of both state variables. To clarify this result, Figure 3.3 depicts sections of the numerical difference between the prices in the presence of storage and the prices in the absence of storage in the economy. Figure 3.4 shows the variability of the price as a function of supply at different levels of storage.

All the results confirm the intuition and the predictions in the theory of storage. Figure 3.1 shows that the storage rate increases with the value of the supply rate and decreases with the storage level. Figures 3.2, 3.3 and 3.4 illustrate that the existence of storage stabilizes the prices. More specifically, if the commodity price is above the natural long-run mean (because supply is low), the existence of storage lowers the prices in relation to the natural price. On the other hand, if the price is below natural long-run mean (because supply is high), the existence

⁸Because of the additive form of our model, we prefer to calculate the standard deviation of the commodity spot price process, dP , rather than the standard deviation of $\frac{dP}{P}$ as in a conventional volatility measure. Accordingly, we name this measure spot price variability instead of spot price volatility.

of storage increases the prices in relation to the natural price. We conclude that storage affects the price dynamics by keeping the prices more stable and closer to the long-run mean, dampening down the slope of the original price function⁹. However, if the supply is high (the price is above the mean) and all the storage capacity has been used, the storage agents are being prevented from storing any further quantity of the commodity from the market and the price falls, behaving as in the case of non-storage. This last result can clearly be observed in Figure 3.3. The spot price variability curves are represented in Figure 3.4, which corroborate the following results: the existence of storage significantly reduces the price variability as a function of supply. Moreover, this reduction is positively related to the inventory level. The exceptions occur when the aggregate storage facility is empty or when the full storage capacity is being used. In these two cases, the variability is equal to 16, which is the same value as it would be observed in a non-storage economy.

⁹The variability of the prices is directly proportional to the slope of the prices as a function of supply. Therefore damping down the slope means reducing the price variability.

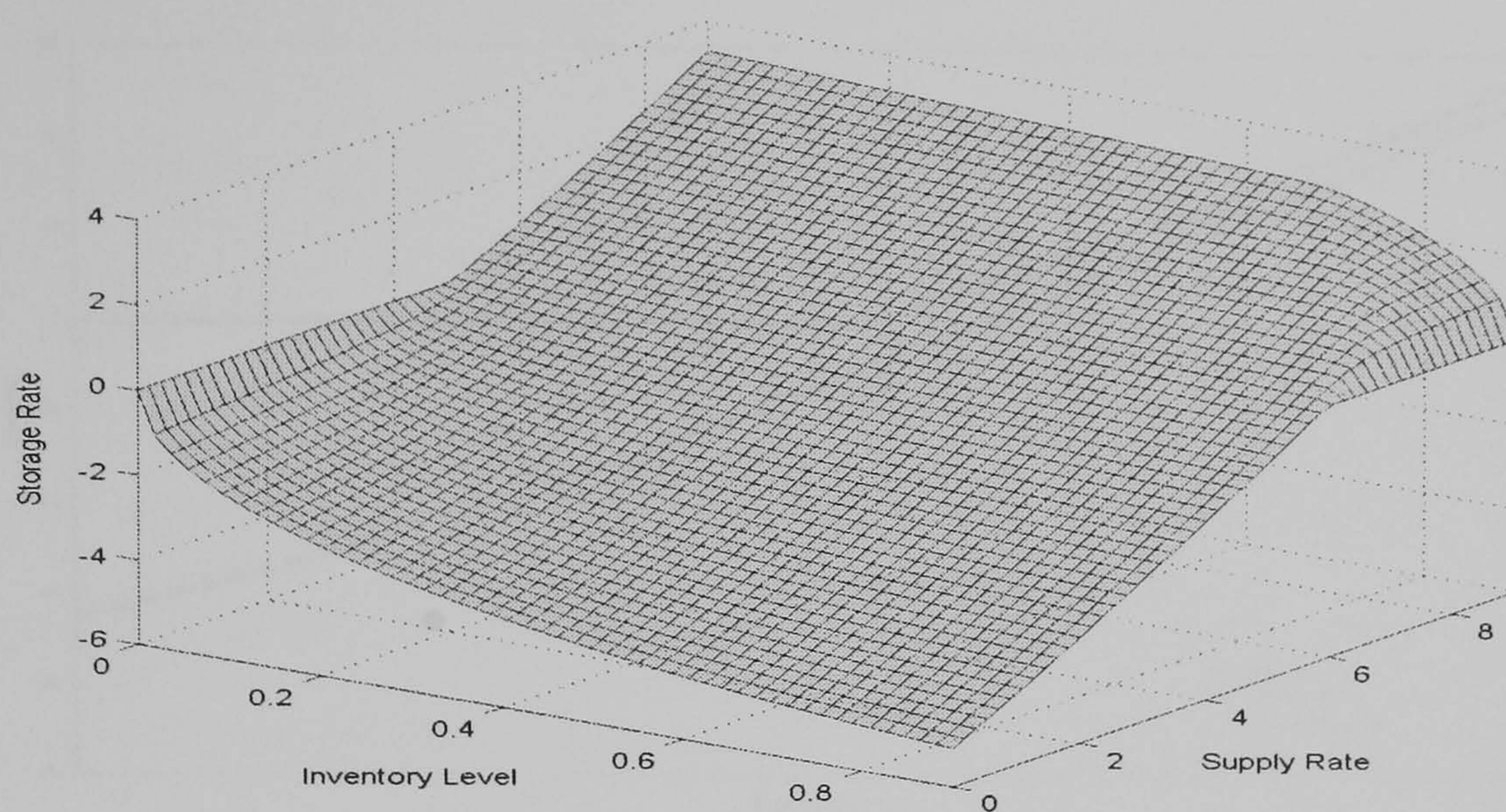


Figure 3.1: Competitive Case - Storage rate, u , as a function of the two state variables inventory level, s , and exogenous rate of supply, z .

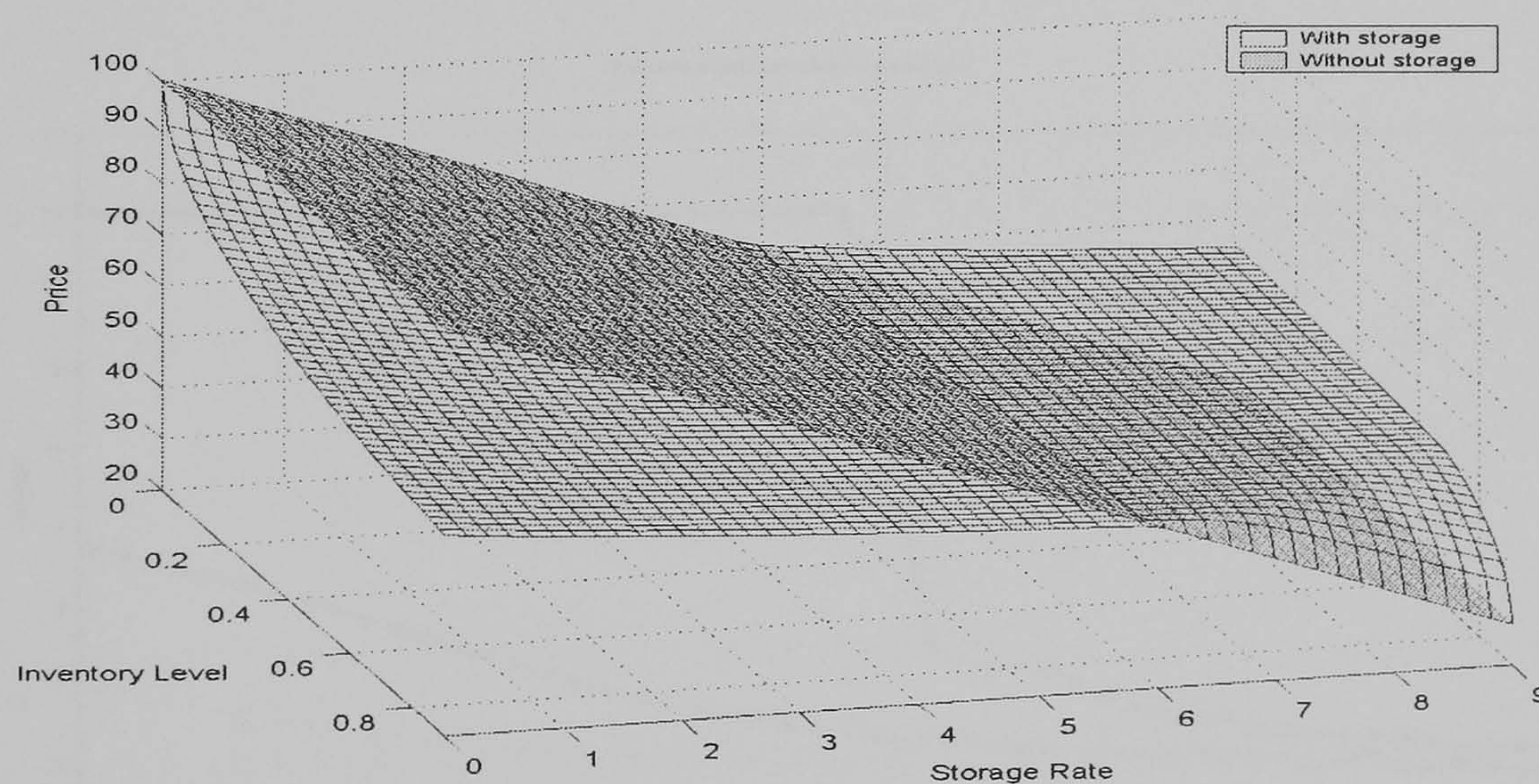


Figure 3.2: Competitive Case - Super-position of two graphs for the prices in the absence and in the presence of storage, respectively. The price is represented as a function of the two state variables inventory level, s , and exogenous rate of supply, z .

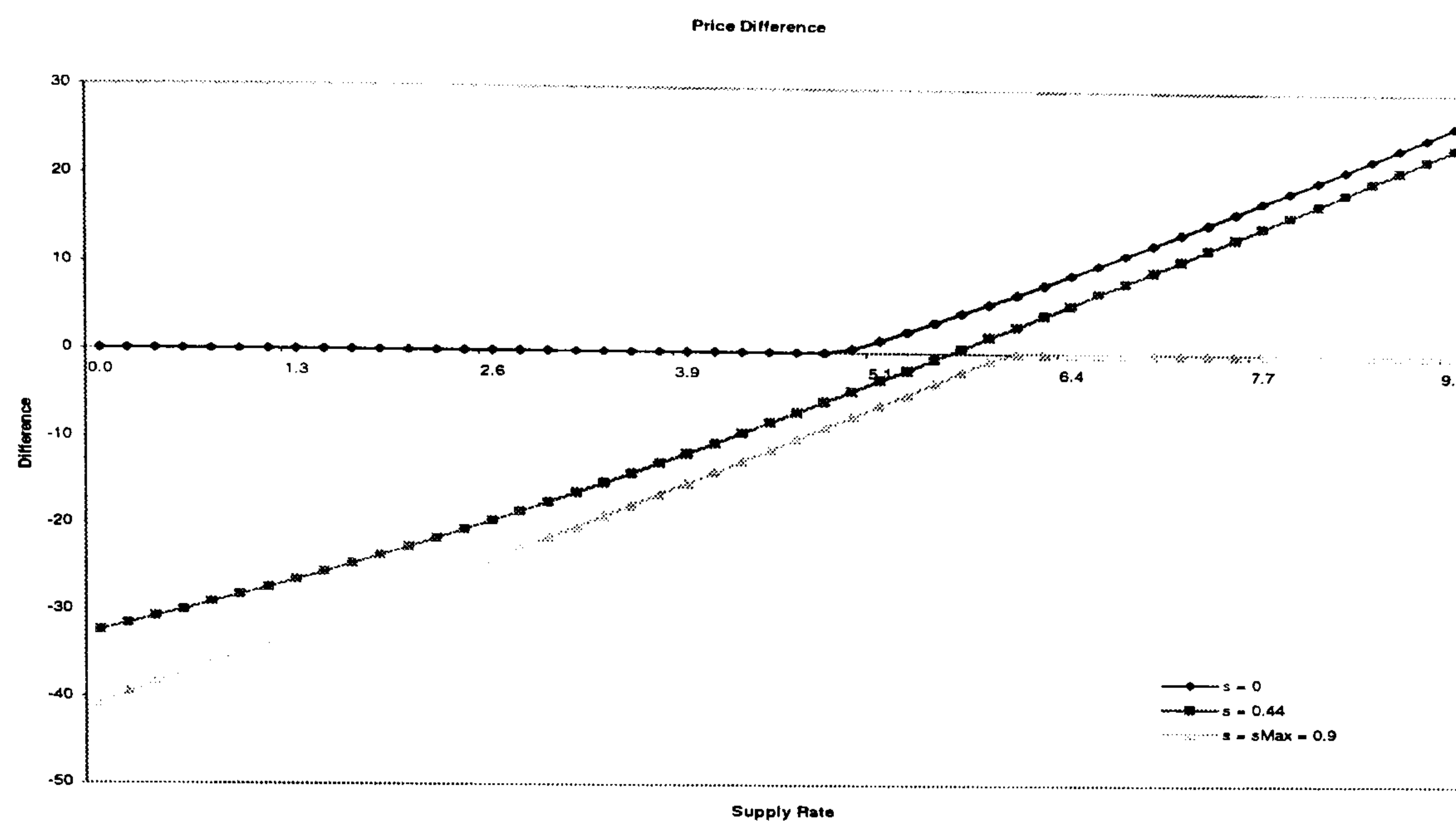


Figure 3.3: Competitive Case - Difference between prices in the presence of storage and prices in the absence of storage as a function of supply rate, at different fixed levels of inventory.

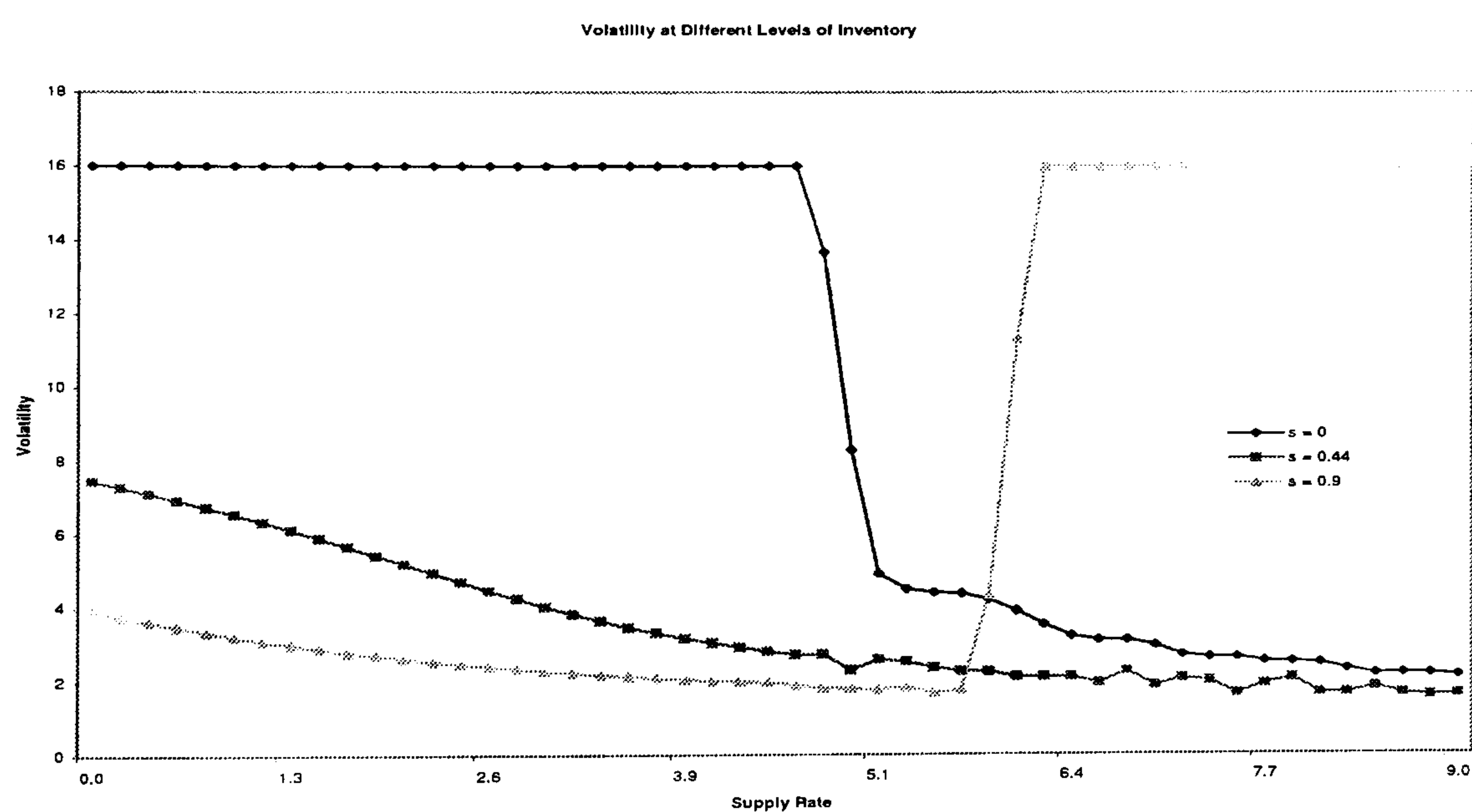


Figure 3.4: Competitive Case - Price variability as a function of supply rate, at different fixed levels of inventory.

3.4.2 Monopolistic Case

Figures 3.5, 3.6, 3.7 and 3.8 represent the equivalent results for the monopolistic case. Figure 3.5 shows that the storage rate increases with the rate of supply and decreases with the level of storage. When supply is high, the storage rate is relatively large and positive. However, when the supply rate is high *and* the inventory level is close to its capacity, the storage rate is forced to be reduced. Similarly to the competitive case, Figures 3.6, 3.7 and 3.8 show that the existence of storage smoothes the price behaviour by comparison with the non-storage case. However, if the inventory level is close to capacity, the stockholder is prevented from buying additional stock, even if it would be optimal to do so. Similarly, when the inventory is empty, the stockholder cannot sell the commodity, even if it was profitable to do so, since commodity short sales are not allowed in a storage economy.

3.4.3 Comparison Between the Competitive and the Monopolistic Markets

A comparison between Figures 3.1 and 3.5 show that the monopolist transacts less than the competitive stockholder at all supply levels. As a result the extent to which the monopolist actions smooth the price smaller than in the competitive case. This is observed by comparing Figures 3.2 and 3.3, 3.4 with figures 3.6, 3.7 and 3.8. Moreover, since the monopolist builds less inventory than the competitive stockholder, the capacity constrain on the storage policy for high supply levels is more prominent in the competitive market than in the monopolistic market.

These results show that the monopolistic stockholder benefits from performing less transactions than the competitive storer, thereby benefiting from a

higher spread between the buying prices and the selling prices. The result of these policy differences is that the monopolist reduces less the variability of the natural commodity spot price behaviour than the competitive one.

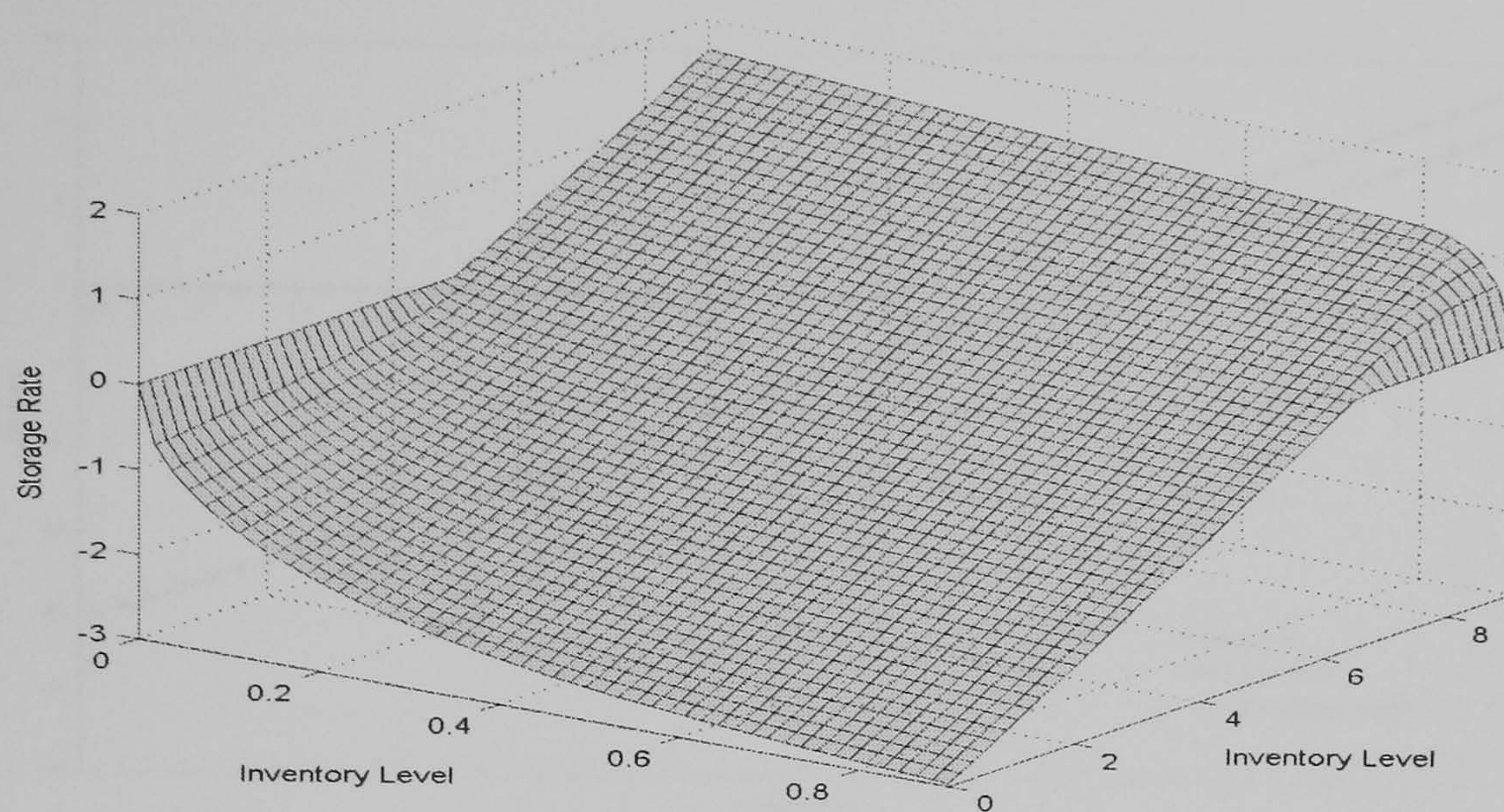


Figure 3.5: Monopolistic Case - Storage rate, u , as a function of the two state variables inventory level, s , and exogenous rate of supply, z .

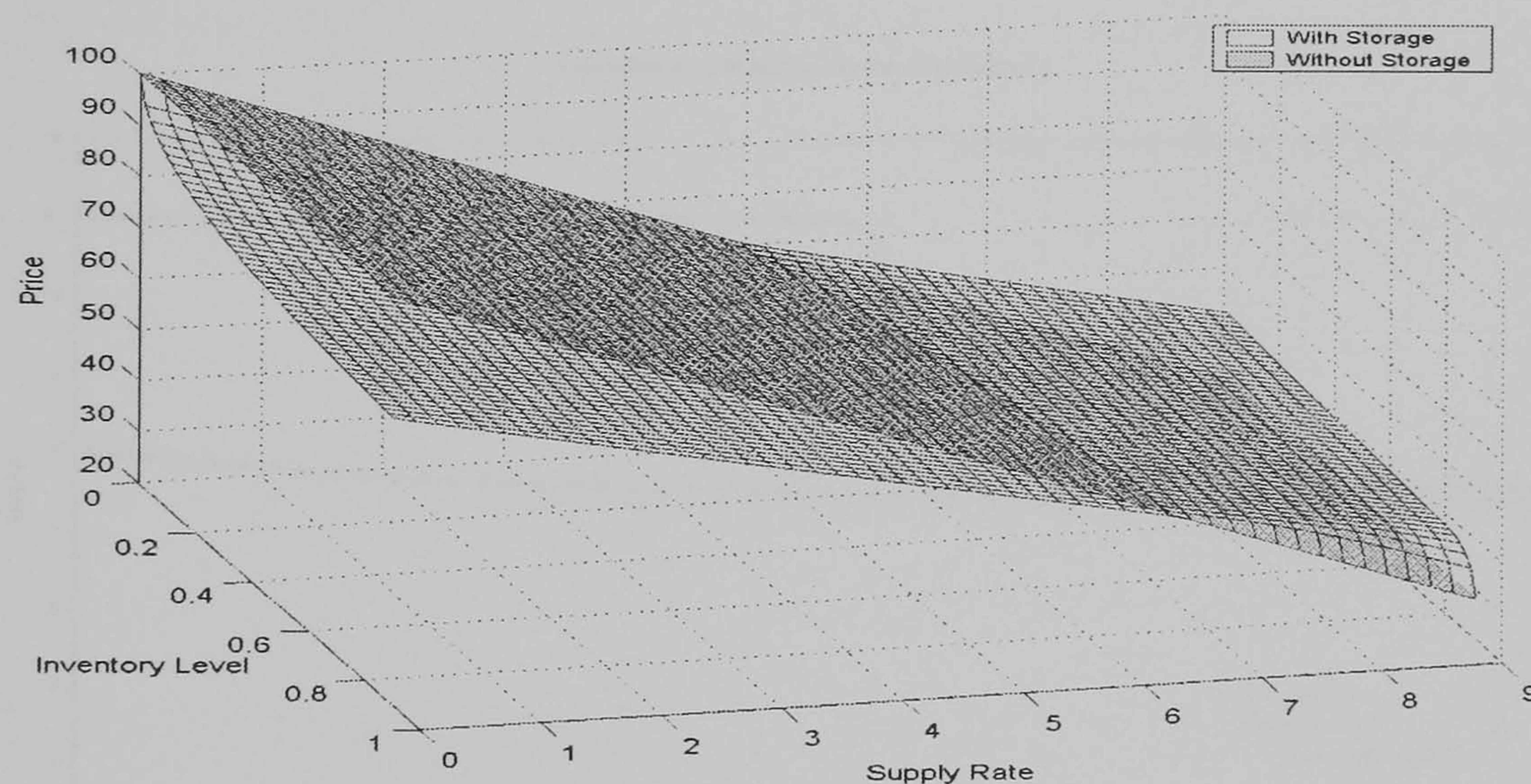


Figure 3.6: Monopolistic Case - Super-position of two graphs for the prices in the absence and in the presence of storage, respectively. The price is represented as a function of the two state variables inventory level, s , and exogenous rate of supply, z .

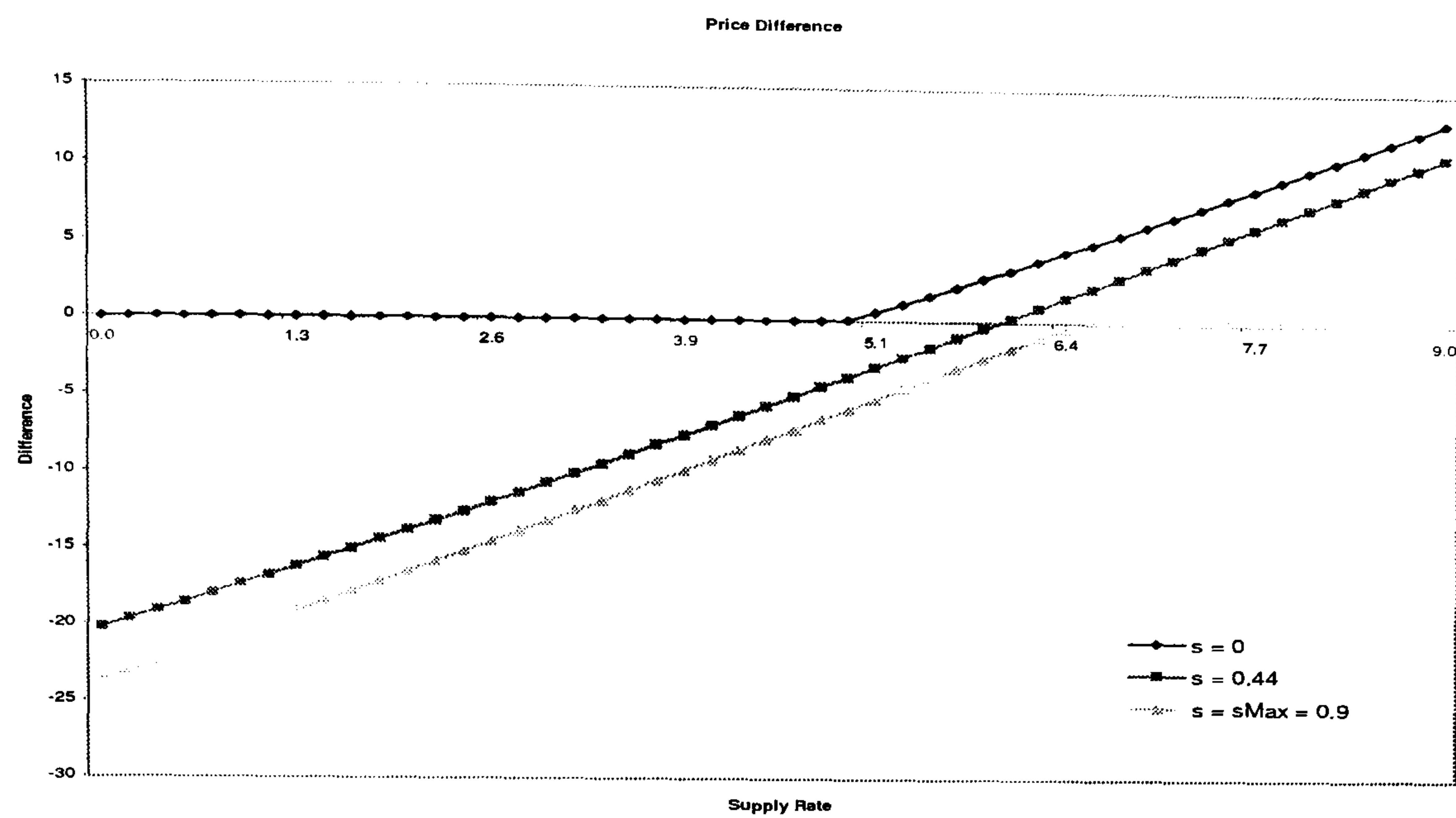


Figure 3.7: Monopolistic Case - Difference between prices in the presence of storage and prices in the absence of storage as a function of supply rate, at different fixed levels of inventory.

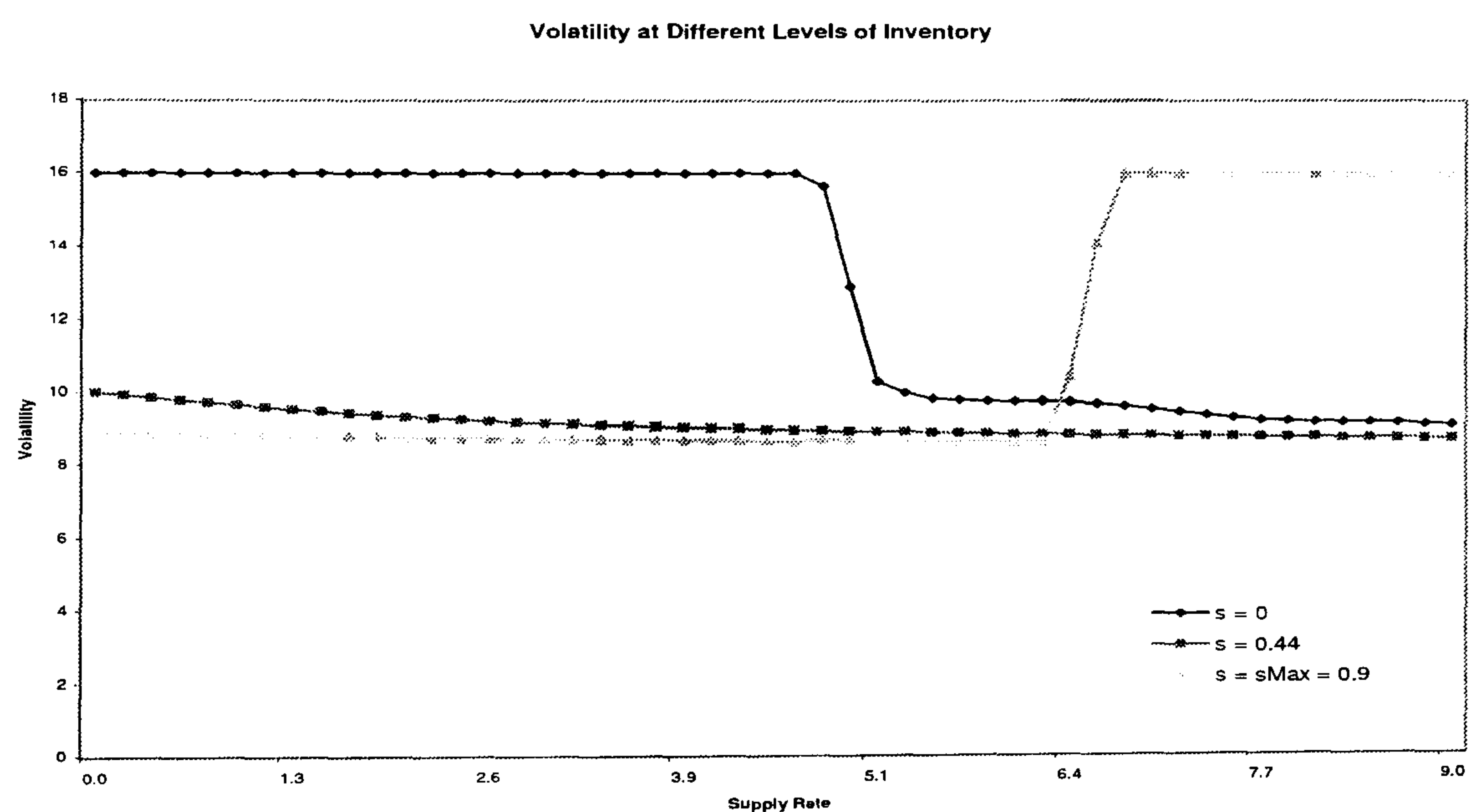


Figure 3.8: Monopolistic Case - Price variability as a function of supply rate, at different fixed levels of inventory.

3.5 Extension of the Model to Include Seasonality

One of the limitations of the model presented above is that it does not take into account the seasonality of commodity prices. As mentioned previously, seasonality is one of the key properties of most commodity prices. In agricultural commodities, the main source of seasonality comes from the supply side since the crops take place on a seasonal basis. In energy markets, such as electricity and natural gas, the source of seasonality is comes from the demand. In particular, residential users create seasonal effects on electricity and natural gas markets, since these are primarily used for heating/cooling purposes.

The model presented in this chapter assumes that the source of uncertainty comes from supply. Therefore, we can include seasonality in this model by adding a seasonal component to the supply. In particular, we consider a seasonal component as a sinusoidal function of period one. Denote by x_t the total rate of supply. Accordingly, x_t is given by:

$$x_t = z_t + c \sin(2\pi t), \quad (3.33)$$

where z_t is the exogenous supply considered previously, which evolves as an O-U stochastic process given by equation (3.1) and c is a constant. Therefore, the transition equation for x_t is the following:

$$dx_t = (\alpha(\bar{z} - z) + 2\pi c \cos(2\pi t)) dt + \sigma dB_t. \quad (3.34)$$

Following the derivation steps described in appendix A, the dynamic programming equation for this problem becomes:

$$-\frac{\partial V(s, z, t)}{\partial t} - H(s, z, V_s, V_z, V_{zz}) = 0, \quad (3.35)$$

where:

$$\begin{aligned}
H(s, z, V_s, V_z, V_{zz}) = & \sup_{u \in [u_{min}, u_{max}]} \{ L(s, z, u, t) + uV_s(s, z, t) + \\
& (\alpha(\bar{z} - z) + 2\pi c \cos(2\pi t)) V_z(s, z, t) + \frac{1}{2}\sigma^2 V_{zz}(s, z, t) - \\
& rV(s, z, t) \}
\end{aligned} \tag{3.36}$$

The only difference between the PDE for the original model, given by equations (3.7) and (3.8), and the PDE for the extended seasonal model described by equations (3.35) and (3.36) is the coefficient of the V_z term. Hence, we can apply the same numerical methods as described in section 3.3 to solve the extended model.

In summary, we showed that it is extremely easy to modify the original formulation of this model to incorporate seasonality. The equilibrium model presented in this chapter is formulated using a general and flexible approach. For this reason, it would be possible to further extend this model by allowing different specifications the microeconomic characteristics of the commodity market without adding significant complexity to the solution. For example, we could consider different shapes of the demand function or define alternative stochastic processes for the supply without adding extra complexity to the solution method. It is this flexibility that allowed us to study and compare the competitive and the monopolistic cases.

3.6 Conclusion

In this chapter we presented a continuous time stochastic structural model suited for non-perishable storable commodity prices, where the source of uncertainty comes from the exogenous supply. This model builds upon the existing discrete time structural models. However, it also takes into account relevant features of reduced form models recently developed in the literature by accounting for the mean-reverting characteristics of spot commodity prices.

This model is formulated as a stochastic dynamic programming problem in continuous time and considers the existence of two state variables: (i) the exogenous stochastic supply, which evolves as a O-U stochastic process and (ii) the endogenous inventory level, which is a fully controllable variable. The decision variable is the rate of storage, which in turn determines the final commodity prices. In order to simplify the numerical computation, we considered a linear inverse demand function in the numerical examples provided. The model is initially formulated under a general framework and later unfolds into two distinct market scenarios - competitive and monopolistic markets.

All the results are in accordance with the theory of storage. The presence of storage in the economy smoothes the spot price behavior by reducing the variability of the natural spot price from the no-storage case. Moreover, the degree of reduction in this variability is positively related to the level of inventory. This smoothing effect is more evident in the case of storage competition than it is in the case of monopolistic storage. This difference results from the observation that the monopolist performs less trading activity than the competitive storer since he benefits from having a greater spread between the buying prices and the sales prices. Another relevant result involves the effect of the storage capacity on the storage policy. In particular, if the storage capacity is fully used (or close to), the stockholders in both economies are not able to respond optimally to price variations. Consequently, the price dynamics will follow the natural price process.

The model presented in this chapter makes several contributions to the current literature. First, it introduces a continuous time structural model that draws on specific microeconomics assumptions of the market environment and establishes a link with the existing reduced form models in the literature. That is, it builds on the structural models but it uses a continuous time framework and includes

the mean reverting characteristics of commodity prices. This latter contribution is particularly relevant since none of the existing structural models has included the mean reversion property of commodity prices. Second, this model is developed under a very flexible framework which allows for different extensions of the model to be adapted to different commodities. For example, this model can be extended to accommodate other type of supply/demand functions. In particular, we explained how seasonality could be included in the model without adding extra complexity to the solution method. Third, we formulated this model for both competitive and monopolistic storage environments and provided a valuable comparison between competitive and monopolistic storage policies and how these differences are reflected in the price dynamics. This comparison is not illustrated in the current storage models, which only consider a competitive storage economy. Since the energy markets have evolved from a monopolistic to a competitive environment in recent years, we stress the importance to analyze both storage economies in order to understand the implications of the market evolution in the price dynamics.

In summary, this model suggests testable hypothesis concerning the dynamics of commodity spot/futures prices, which will help us to develop the reduced form models presented in Chapters 5 and 6.

One of the directions for further work should include the extension of the analysis to encompass non-linear demand functions, which is a more realistic assumption than having a linear demand function. Another direction of future work is the numerical implementation and analysis of the seasonal version of this model as suggested in section 3.5. Since most of commodity prices have seasonality, the study of the seasonal effect in the price dynamics is significant. It would also be useful to study the case where the supply includes jumps since one of the energy price characteristic is the occurrence of occasional spikes. This could easily be

included in the model by adding a Poisson process to the supply stochastic process in the spirit of the jump diffusion process presented by Merton (1976) [66]. Allowing capacity investment is also worth exploring. The integration of a real options model like that of Dixit and Pindyck (1994) [28] with the richer environment of this model is an interesting, and certainly challenging, possibility. Another direction is the development of a steady state version of the model presented in this chapter. Although this seems to be extremely difficult to obtain under realistic assumptions, it would be interesting to find a method to study the steady state case directly.

Chapter 4

A Structural Model for the Price Dynamics: Analysis of the Forward Curve

4.1 Introduction

In this chapter we implement and analyze the forward curve corresponding to the structural commodity price model presented in Chapter 3 and compute the present value of a storage facility for both competitive and monopolistic markets. Using the steady-state optimal storage policy developed in the previous chapter, we construct a trinomial tree for the commodity prices and the corresponding forward curve that evolves by computing at each node the optimal combinations of inventory level and exogenous supply rate.

Although inspired by Hull and White (1993a) [55] our numerical procedure is significantly different. In their paper, the authors extend the standard binomial tree procedure described by Cox, Ross and Rubinstein (1979) [21] to value some

types of path-dependent options such as European and American options on the arithmetic average price of an asset, the so-called Asian options. This type of options is difficult to price since the number of average stock prices realized between zero and a node can be very large. Hull and White suggest to price the option only for a predetermined representative set of values for the path-dependent function - the average, in this case - and interpolate to calculate option value as required. The value of the option for other values of the path-dependent function is computed from the known values by interpolation as required. This approach does not constrain the number of the values at which the path-dependent function is calculated.

In this study, we build a trinomial tree for the Ornstein-Uhlenbeck (O-U) process that describes the stochastic supply in the model presented in the previous chapter applying standard methods as described by Hull and White (1993b, 1994) [56, 57]. At time zero, we assume predetermined values for the supply rate and the level of storage. The storage levels for each node in the tree evolve from this starting point by computing the optimal rate of storage for each combination of supply and inventory. This optimal rate is calculated by interpolation using the steady optimal storage policy values obtained from the computational implementation of the structural model in Chapter 3. We can think of the calculation of the optimal storage policy as the equivalent to the calculation of the path-dependent function in Hull and White (1993a) [55]. Similarly, as the time evolves in the tree, the possible number of storage levels for each node representing the supply rate increases very rapidly. However, unlike Hull and White, we do not choose predetermined values of storage levels but impose a maximum number of these at each node. When the number of combinations is above a certain predetermined value, we merge the storage values that are associated with a particular supply value

into a predetermined (smaller) number of values. The reduction in the number of nodes is subtle to avoid a large loss of information. The commodity price is calculated for each existing combination of supply and storage. With the use of these values the current forward prices are computed as the expectation of the spot price at each maturity of the forward curve, conditional on the initial values of inventory and supply.

The numerical procedure presented in this chapter enables the valuation of forward prices and other derivatives using an equilibrium model for the commodity spot price. The current commodity structural models restrict the analysis to the properties of the spot prices as a function of the fundamental state variables in the model and do not extend it to evaluate forward curves or other derivatives. The exception is the paper presented by Routledge, Seppi and Spatt (2000) [78], which present a study of equilibrium forward curves for commodities but their analysis has limitations. They present an equilibrium model that builds on the main stream of the literature in structural models (Deaton and Laroque, 1992; 1996 [24, 25] and Chambers and Bailey, 1992 [12]) for commodity prices where the source of uncertainty is the stochastic demand. The demand shocks are represented by a finite, irreducible, m -state Markov process where $m \geq 2$ with a transition probabilities defined by a matrix. This representation makes it difficult to analyze the forward curves generated by their model for a realistic large number of demand states. The authors therefore limit their analysis to the case where the stochastic demand can only take two values - high and low - which is unrealistic. Moreover, the extension of their results to a more general Markov process is difficult to obtain. On the other hand, the numerical method and the corresponding analysis presented in this chapter is relatively easy to implement and to generalize to any initial combinations of supply and storage values.

We analyze and compare different types of forward and convenience yield curves obtained by varying the initial values of the inventory level in the model. Although we only illustrate the results for a unique initial value for the supply rate, the generalization for other values of initial supply is straightforward.

The remaining of this chapter is organized as follows. Section 4.2 describes the lattice model. Section 4.3 computes and presents the numerical results of the forward curve and the convenience yield. Section 4.4 concludes.

4.2 Description of the Lattice Model

4.2.1 The Branching Process

The tree that represents the evolution of commodity prices is the result of two main steps. The first is the construction of the tree representing the O-U process that describes the supply rate process as described in Chapter 3. The second is the calculation of the optimal storage levels which result from the application of the steady-state storage policy developed in the previous chapter. Each combination of storage and supply yields a unique commodity price with a certain probability.

As described in Chapter 3, the stochastic process for the exogenous supply rate, $z(t)$, is the following:

$$dz_t = \alpha(\bar{z} - z_t)dt + \sigma dB_t, \quad t \geq 0 \quad (4.1)$$

where:

- α is the speed of mean reversion;
- \bar{z} is the long-run mean, that is, the level to which z reverts as t goes to infinity;

- σ is the (constant) volatility;
- B_t is a standard Wiener process.

The tree is constructed using the standard methods described by Hull and White (1993b, 1994) [56, 57], which is described in detail in appendix C. Denote by Δt the length of the time-steps and Δz the length of the z -steps. At each time-step, z takes the value $z_0 + j\Delta z$, where j can either be positive or negative and z_0 is the initial value. Denote by (i, j) the node at which $t = i\Delta t$ and $z = j\Delta z$. The trinomial branching process can take alternative forms, that is, a normal branching process where we can move up by Δx , stay the same and move down by Δx . We use the step size $\Delta z = \sigma\sqrt{3\Delta t}$ ¹ and define $pr_{i,j}^u$, $pr_{i,j}^m$ and $pr_{i,j}^d$ as the probabilities of the highest, middle and lowest branches emanating from node (i, j) . As described in the appendix C, the probabilities are chosen to match the expected change and variance in z over the next interval Δt and must sum up to unity. The probabilities are given by:

$$\begin{aligned} pr_{i,j}^u &= \frac{\sigma^2\Delta t + \eta^2}{2\Delta x^2} + \frac{\eta}{2\Delta x} \\ pr_{i,j}^m &= 1 - \frac{\sigma^2\Delta t + \eta^2}{\Delta x^2} \\ pr_{i,j}^d &= \frac{\sigma^2\Delta t + \eta^2}{2\Delta x^2} - \frac{\eta}{2\Delta x} \end{aligned} \quad (4.2)$$

where $\eta = \mu_{i,j} + (j - k)\Delta x$ and $k = j$ for a normal branching process, $k = j + 1$ when z is currently low and $k = j - 1$ when z is currently high in the tree².

The other state variable in our structural model, the aggregate storage level, s , is a fully controllable endogenous state variable and satisfies:

$$ds = u(s, z)dt, \quad s \geq 0 \quad (4.3)$$

¹Provided that Δx is within the range $\sigma\sqrt{3\Delta t}/2$ to $2\sigma\sqrt{\Delta t}$, the probabilities are always between 0 and 1 (Hull and White (1993b) [56]).

²For further explanation see appendix C.

where u represents the rate of storage and is the decision variable in our problem. At each time t , the rate at which the commodity is stored depends on the amount already in storage, s_t , and on the exogenous supply, z_t , as described in Chapter 3. Accordingly, at time t , for each combination of inventory level, s_t , and exogenous supply z_t , there exists an optimal storage rate, $u^*(s_t, z_t)$. This value is obtained through interpolation³ using the long-run optimal storage policy, $u^*(s, z)$ obtained numerically in Chapter 3⁴.

According to equation (4.3), given the storage level at time $t - 1$ and the optimal storage rate u^* , the inventory level at time t is given by:

$$s_t = s_{t-\Delta t} + u^*(s_{t-\Delta t}, z_{t-\Delta t})\Delta t \quad (4.4)$$

Given the Markovian structure of both the exogenous supply rate and the storage process, it is always possible to compute $(s_{t+\Delta t}, z_{t+\Delta t})$ from (s_t, z_t) .

We denote the k^{th} value of s at node (i, j) by $s_{i,j,k}$, for $k = 1, \dots, k_{i,j}$ where $k_{i,j}$ is the number of possible (s, z) combinations at node (i, j) . As illustrated in Figure 4.1, for $t \geq 2$, for each value of $z_{i,j}$ in the tree we have k values of inventory levels. Thus, the number of possible combinations of inventory level and supply rate, (s, z) at each node grows rapidly. Let (i, j_{max}) and (i, j_{min}) denote the upper and the lower node at time $t = i\Delta t$. The number of possible combinations (s, z)

³We use the local Shepard interpolation method described in Chapter 9 of Engeln-Mullges and Uhlig (1996) [34]. This numerical procedure is described in appendix D.

⁴We are calculating u^* by interpolating the steady-state storage policy calculated in Chapter 3, that is, when $T \rightarrow \infty$. Therefore u^* does not depend on time t .

at any internal node (i, j) is given by:

$$\begin{aligned}
k_{i,j} &= k_{i-1,j-1} + k_{i-1,j} + k_{i-1,j+1}, & \text{if } j_{min} + 2 < j < j_{max} - 2 \\
k_{i,j} &= k_{i-1,j-1} + k_{i-1,j}, & \text{if } j = j_{max} - 1 \\
k_{i,j} &= k_{i-1,j+1} + k_{i-1,j}, & \text{if } j = j_{min} + 1 \\
k_{i,j} &= k_{i-1,j-1}, & \text{if } j = j_{max} \\
k_{i,j} &= k_{i-1,j+1}, & \text{if } j = j_{min}
\end{aligned} \tag{4.5}$$

At the edge of the tree, the equations are slightly different due to the mean reverting nature of the exogenous supply process, in order to avoid negative probabilities, as described in appendix C. The calculation of the number of nodes at these edges follows similar reasoning and is not described here for brevity.

Denote by $p_{i,j,k}$, for $k = 1 : k_{i,j}$ the probability of the combination $(s_{i,j,k}, z_{i,j})$ occur in node (i, j) . Consider for example node (i, j) , for $j_{min} + 2 < j < j_{max} - 2$. Additionally, suppose that $(s_{i,j,k_1}, z_{i,j})$, for $1 \leq k_1 \leq k_{i,j}$, with k_1 being fixed arises from $(s_{i-1,j-1,k_0}, z_{i-1,j-1})$, $1 \leq k_0 \leq k_{i-1,j-1}$, with k_0 being fixed. Then, the corresponding probability for pr_{i,j,k_1} is given by:

$$pr_{i,j,k_1} = pr_{i-1,j-1,k_0} * pr_{i-1,j-1}^u \tag{4.6}$$

where $pr_{i-1,j-1}^u$ is given by equation (4.2). The other cases follow trivially taking into account the geometry of the trinomial tree and equations (4.2) and (4.5).

As we saw in Chapter 3, the spot price of the commodity is given by:

$$p(q) = p(z - u^*) \tag{4.7}$$

where z is the rate of exogenous supply evolving according to equation (4.1), and u^* is the optimal storage rate resulting from the optimal storage policy in the long run as described in the previous chapter. As mentioned above, for each specific combination (s, z) we calculate u^* by interpolating the values of the optimal

storage policy designed in the previous chapter. Therefore, for each combination $(s_{i,j,k}, z_{i,j})$, $k = 1, \dots, k_{i,j}$, where $k_{i,j}$ is the number of possible (s, z) combinations at node (i, j) , there is an optimal storage rate associated with it, $u^*(i, j, k)$. Clearly, the probability associated with this optimal storage rate is the same as the probability associated with the combination of $(s_{i,j,k}, z_{i,j})$. This, in turn, also implies that this same probability is associated with the resulting spot price, $p(s_{i,j,k}, z_{i,j})$. This allows us to calculate the resulting price expectation at each time in the tree. Since we are working in the risk-neutral measure and the interest rate is non-stochastic we have that:

$$F_{t,T} = E_t[p_T], \quad (4.8)$$

where E_t denotes the conditional expectation under the risk neutral probabilities given the information at time t .

Note that the probability attributed to a combination of (s, z) is calculated forward as the tree evolves. This enables us to compute the expected value of the spot price at each time in the tree without needing to calculate the expectation backwards. In this particular problem a forward calculation is simpler due to the merging process of nodes explained in the next section.

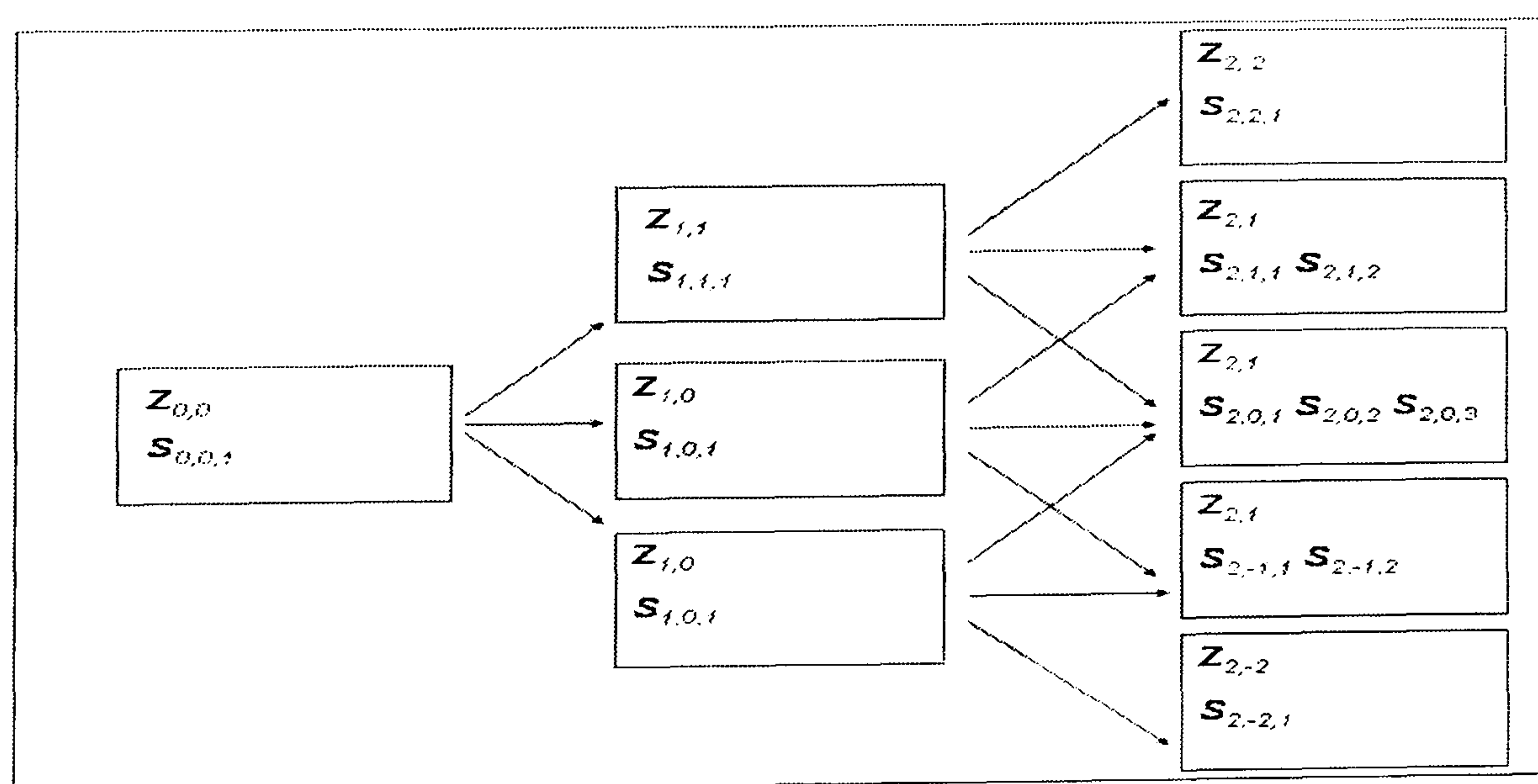


Figure 4.1: Part of the tree for computing the forward curve.

4.2.2 The Merging Process

The numerical method described in the previous section implies that the number of possible combinations of the state variables, (s, z) , grows very quickly with the size of the tree, becoming computationally inefficient. To avoid this problem we place a constraint on the number of the combinations (s, z) at each node of the basic tree that evolves exogenous supply process given by equation (4.1). In other words, if the number of combinations (s, z) in a node exceeds say l then we merge these combinations into l_{New} combinations such that $l_{New} < l$. Before a merger takes place we first sort the storage levels to be merged by increasing order, starting with the smallest. This ensures that the mergers are effectuated between adjacent values of storage levels. This merging process is done using linear interpolation weighted by the corresponding probabilities, as described by equation (4.9) below. Note that we only merge the values of storage levels, s , while the corresponding value of z remains the same. Denote by $s_{i,j,k_{New}}$ the storage level that results from the merger of two nodes and by s_{i,j,k_0} and s_{i,j,k_1} the two nodes to be merged. The resulting node is given by⁵:

$$s_{i,j,k_{New}} = \frac{pr_{i,j,k_0} * s_{i,j,k_0} + pr_{i,j,k_1} * s_{i,j,k_1}}{pr_{i,j,k_0} + pr_{i,j,k_1}} \quad (4.9)$$

the corresponding probability is the sum of the two probabilities of the corresponding nodes, that is:

$$pr_{i,j,k_{New}} = pr_{i,j,k_0} + pr_{i,j,k_1} \quad (4.10)$$

This process ensures consistency in the calculation of the expectation as described by equation (4.8) above. This process is repeated every time the number of combinations (s, z) , m , is greater than a maximum of l . In this case, these combinations are merged in a predetermined number of nodes, $l_{New} < l$, according

⁵Here we consider only two nodes to be merged for simplicity. However, this can be applied to an arbitrary number of nodes.

to the process described above. This ensures that the number of (s, z) does not grow beyond a certain limit. Note also that the reduction in the number of nodes involved in a merger should be subtle in order to keep accuracy in the resulting calculations.

4.3 Calculation of the Forward Curve and the Convenience Yield

Since we are working in the risk-neutral measure and the interest rate is non-stochastic, the forward curve is calculated using the expectation relationship between forward prices and spot prices as described by equation (4.8) above. Using this relationship we construct the forward curve starting at time $t = 0$ for the period of length T , conditional on a particular initial combination of exogenous supply rate and storage level (s_0, z_0) .

The calculation of the convenience yield relies on the well known relationship between the futures and the spot price of a commodity when the interest rate and the convenience yield are deterministic. If the amount of storage costs incurred between t and $t + dt$ is known and has a present value C at time t the convenience yield, δ , is defined as:

$$F_{t+\Delta t} = (p_t + C)e^{(r-\delta)\Delta t} \quad (4.11)$$

Based on this relationship, we calculate the annualized convenience yield for the time interval between t and $t + \Delta t$ by using pairs of adjacent maturities futures contracts according to the following formula:

$$\delta_{t,t+\Delta t} = r - \frac{1}{\Delta t} \ln \left(\frac{F_{t+\Delta t}}{F_t + C} \right) \quad (4.12)$$

A similar definition for the convenience yield is also used by Gibson and Schwartz (1990) [42].

4.3.1 Results

This section presents and analyzes the commodity forward curves, which are generated by the application of the steady state storage policies presented in Chapter 3.

Table 4.1 displays the time to maturity period, T , the time-step dt , the maximum number of combinations (s, z) allowed at each node of the tree, l , and the number of new combinations (s, z) after the merge takes place, l_{new} . We keep the reduction in the number of the nodes subtle in order to avoid a significant loss of information. Although we implement the tree using a time-step of 0.005, the results plotted in the figures correspond to sample time-intervals of 0.1. The annualized convenience yield is calculated according to equation (4.12) using two forward prices with consecutive maturities which differ by a time interval of 0.1. The parameter values used in the computation of the tree are displayed in Tables 4.2 and 4.3. Note that the values used here are the same as the ones used in Chapter 3, with the exception of the supply rate limits, z_{Min} and z_{Max} , the marginal cost of storage, k , and the total storage capacity, s_{Max} . The supply rate limits are different because they are induced by the trinomial tree that represents the O-U stochastic process for the supply, starting at $z_0 = 4.5^6$ with time-step as above and space-step $dz = \sigma\sqrt{3dt}$. Additionally, we consider a storage capacity considerably greater than the value used in the numerical implementation in the previous chapter to allow for a larger range of initial inventory levels. Finally and without loss of generality, we consider a marginal cost of storage equal to zero to avoid adding further numerical approximations in the calculation of the convenience yield. Specifically, the calculation of the storage costs incurred at each period of time involves the calculation of the total amount of storage at each period of time.

⁶Although here we only present the case where the initial supply rate $z_0 = 4.5$, the results for different initial values of supply follow by analogy.

This calculation, in turn, would be affected by numerical approximation resulting from the interpolations and the merging processes that occur. Besides, keeping the storage costs equal to zero does not modify the results qualitatively.

T	dt	l	l_{new}
9	0.005	30	20

Table 4.1: T represents the maximum time to maturity considered, dt represents the time-step, l represents maximum number of combinations (s, z) allowed at each node of the tree, and l_{new} is the new number of (s, z) combinations after the merging takes place.

a	b	α	σ	\bar{z}	k	r
100	8	6.00	2.00	4.50	0	0.05

Table 4.2: Value of the parameters used to implement the tree for both competitive and monopolistic markets.

z_{Min}	z_{Max}	s_{Max}
0.34	8.66	4.5

Table 4.3: z_{Min} and z_{Max} represent the lower and upper values of the grid for the exogenous supply rate, z . s_{Max} represents the storage capacity.

Figures 4.2 and 4.4 represent the evolution of the forward curve when the initial storage level is equal to zero for the competitive and the monopolistic storage markets, respectively. Not surprisingly, both forward curves are in backwardation since null inventory levels at time zero reflect the possibility of commodity shortages

during the life of the forward contracts, inducing positive convenience yields. In the long-run, both curves move towards a state-independent (constant), long-term forward price, F_∞ , which is equal to the long-run mean natural price⁷ $\bar{P} = p(\bar{z}) = a - b\bar{z} = 64$ where \bar{z} is the long-run mean of the Ornstein-Uhlenbeck stochastic process for the supply rate. This reflects the steady state equilibrium of both storage economies in which the expected total amount of commodity sold is equal to the expected total amount of the commodity bought by the (aggregate) storer. This phenomenon can be observed in Figures ?? and ?? of the next section and will be discussed in more detail later. Figures 4.3 and 4.5 show that the corresponding convenience yield curves decrease with time, matching the shape of the forward curves. In particular, the convenience yield is at its maximum when the storage is empty and decreases convexly as the aggregate inventory increases with time, becoming equal to the riskless interest rate in the steady state long run. This is consistent with the predictions of the theory of storage which states that the convenience yield is a convex function of the aggregate inventory, that is, the convenience yield declines at a decreasing rate as the level of inventory increases.

Each of the Figures 4.6 and 4.8 represent a series of three forward curves for the competitive and the monopolistic markets, respectively. Each of the forward curves displayed correspond to initial inventory levels $s_0 = 1.125, 2.25$ and 4.5 respectively. The evolution of the corresponding convenience yield curves is displayed in Figures 4.7 and 4.9. All the curves are in contango, as expected, since the commodity price at time zero is smaller than the long-run natural commodity price. We also observe that the smaller the initial inventory level is, the greater the initial commodity spot price is. This is due to the fact that the total availability of the commodity decreases. This is also reflected in the length of time at which each of the curves remain in contango. That is, the smaller the initial commodity

⁷The natural price is the price in the absence of storage.

price is the longer the forward curve will remain in contango until it reaches the unconditional forward price $F_\infty = \bar{P}$. This is observed because the slope of the forward curves is (approximately) the same⁸ (within each of the competitive or monopolistic economies) independently of the initial level of storage.

Comparing the forward curves between the competitive and monopolistic markets we note that the slope of the competitive forward curves is greater than the slope of the monopolistic forward curves. This is also verified by the corresponding values of the convenience yield observed within each market. The convenience yield observed in the competitive market is (approximately) zero when the market is in contango. On the other hand, the convenience yield observed in the monopolistic market is positive (but small). This means that the annualized futures returns given by $\ln\left(\frac{F_{t+1}}{F_t}\right)$ is equal to the annualized risk free interest rate when storage is competitive. In contrast, the annualized futures returns are smaller than the interest rate when storage is monopolistic. This implies that the monopolistic storer has a positive benefit from holding inventory explained by the convenience yield. This positive value is a result from the monopolistic storage policy. In particular, the monopolist restricts the quantity he buys since this strategy will guarantee him profitable spreads between the prices at which he buys and sells the commodity. Although the monopolist trades less than the competitive stockholder, these spreads guarantee him greater cash-flows than what we would get following the competitive trading strategy.

In summary, the commodity forward curves take two fundamental shapes depending on whether the initial commodity price is below or above the state-independent long-term forward price, $F_\infty = \bar{P}$. Specifically, if the commodity price is less than the long-run forward price, the curve will be in contango otherwise it will

⁸Note that the results presented are affected by numerical approximation and errors due to the successive interpolations to calculate the optimal storage policy for all the (s, z) combinations in the tree and to the merging process. Therefore the results are affected by some noise.

be in backwardation. In the example provided in this chapter the initial inventory is equal to the long-run average supply, \bar{z} . In this case we observe the following two shapes: (i) when the initial inventory is zero, the forward curve is downward sloping (backwardation) for some time and declines towards the steady state long-term forward price, and (ii) when the initial inventory is positive the forward curve is upward sloping (contango) for some time and rises towards the steady state forward price. Moreover, the amount of time the curve remains in contango is positively related to the initial inventory level. In any case, the forward curve tends to the long-run forward price, $F_\infty = \bar{P}$. These results are consistent with the theory of storage and with the properties inherent to the structural models in the literature and in particular with the forward curve analysis in Routledge, Seppi and Spatt (2000) [78]. These authors assume that the source of uncertainty comes from the demand, where the shocks are modelled by a 2-state Markov process, a *high* demand state and a *low* demand state. They assume an initial low (or zero) inventory level and observe the two following forward curve shape: (i) the curve is upward sloping when the demand state is *low*, which correspond to a low initial spot price and (ii) the curve is downward sloping when the demand is *high*, which corresponds to a high initial spot price. In both cases, the forward curve eventually becomes equal to the long-term forward price, F_∞ .

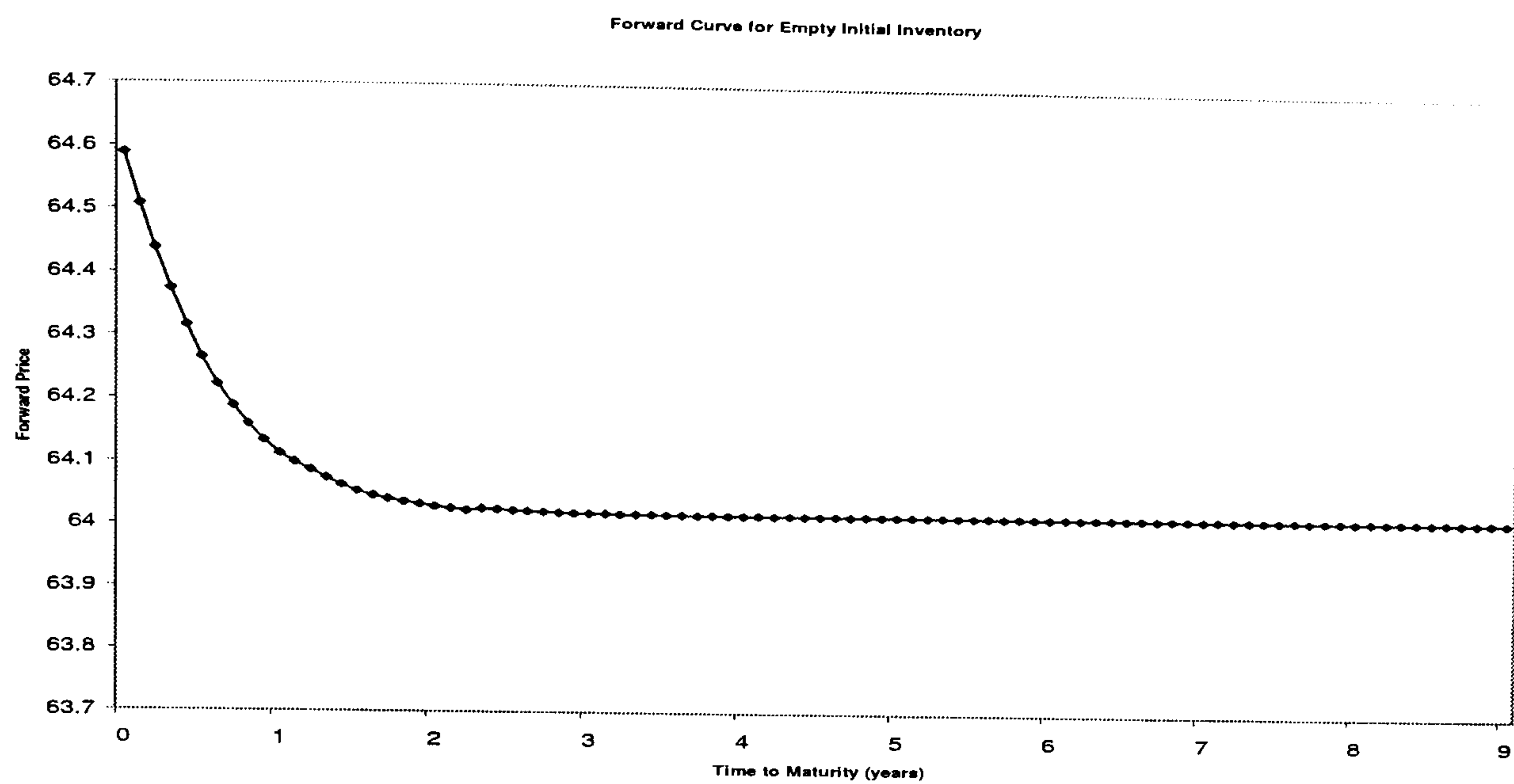


Figure 4.2: Competitive case: Evolution of the forward curve when the initial supply rate is 4.5 and the initial inventory is empty.

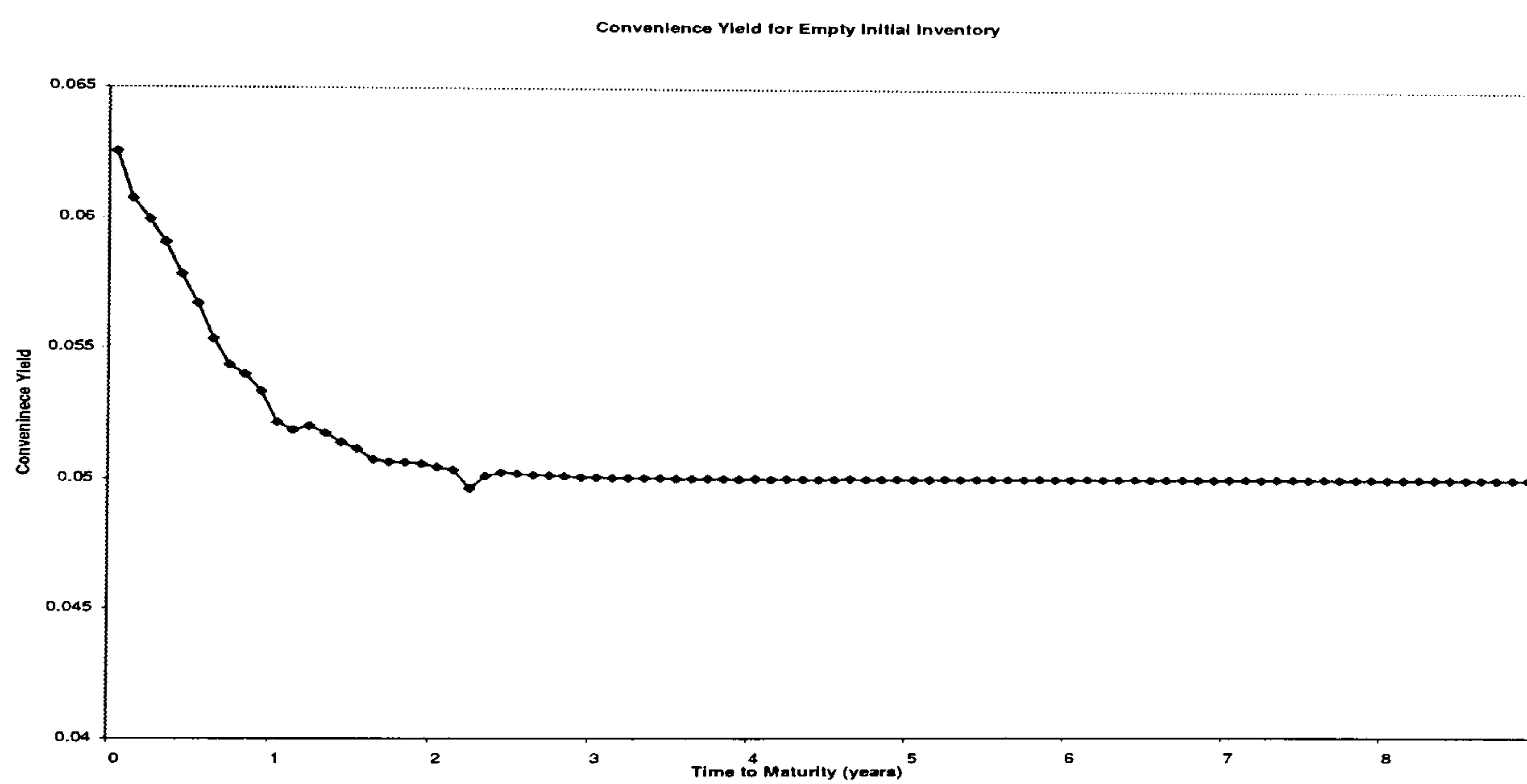


Figure 4.3: Competitive case: Evolution of the convenience yield curve when the initial supply rate is 4.5 and the initial inventory is empty.

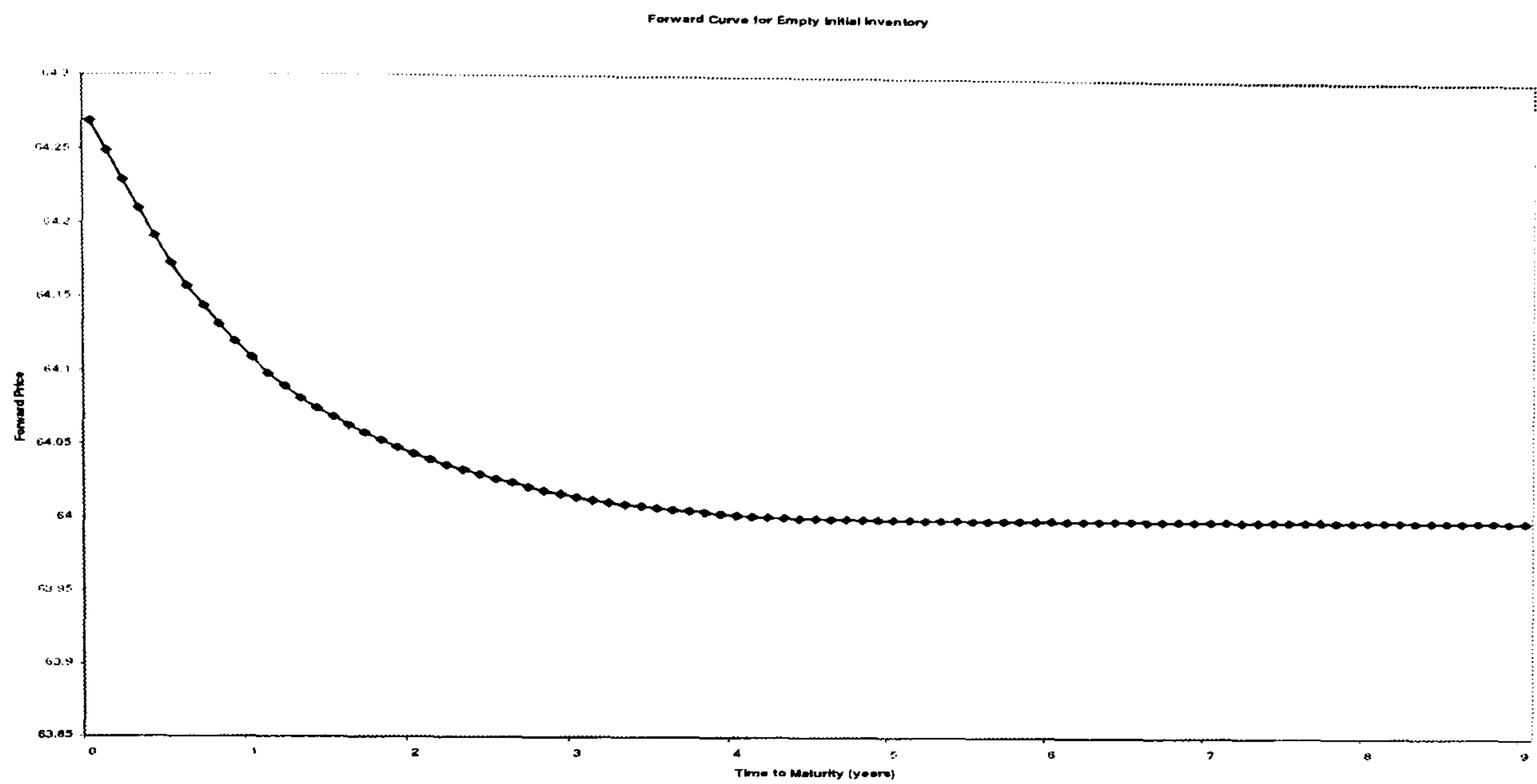


Figure 4.4: Monopolistic case: Evolution of the forward curve when the initial supply rate is 4.5 and the initial inventory is empty.

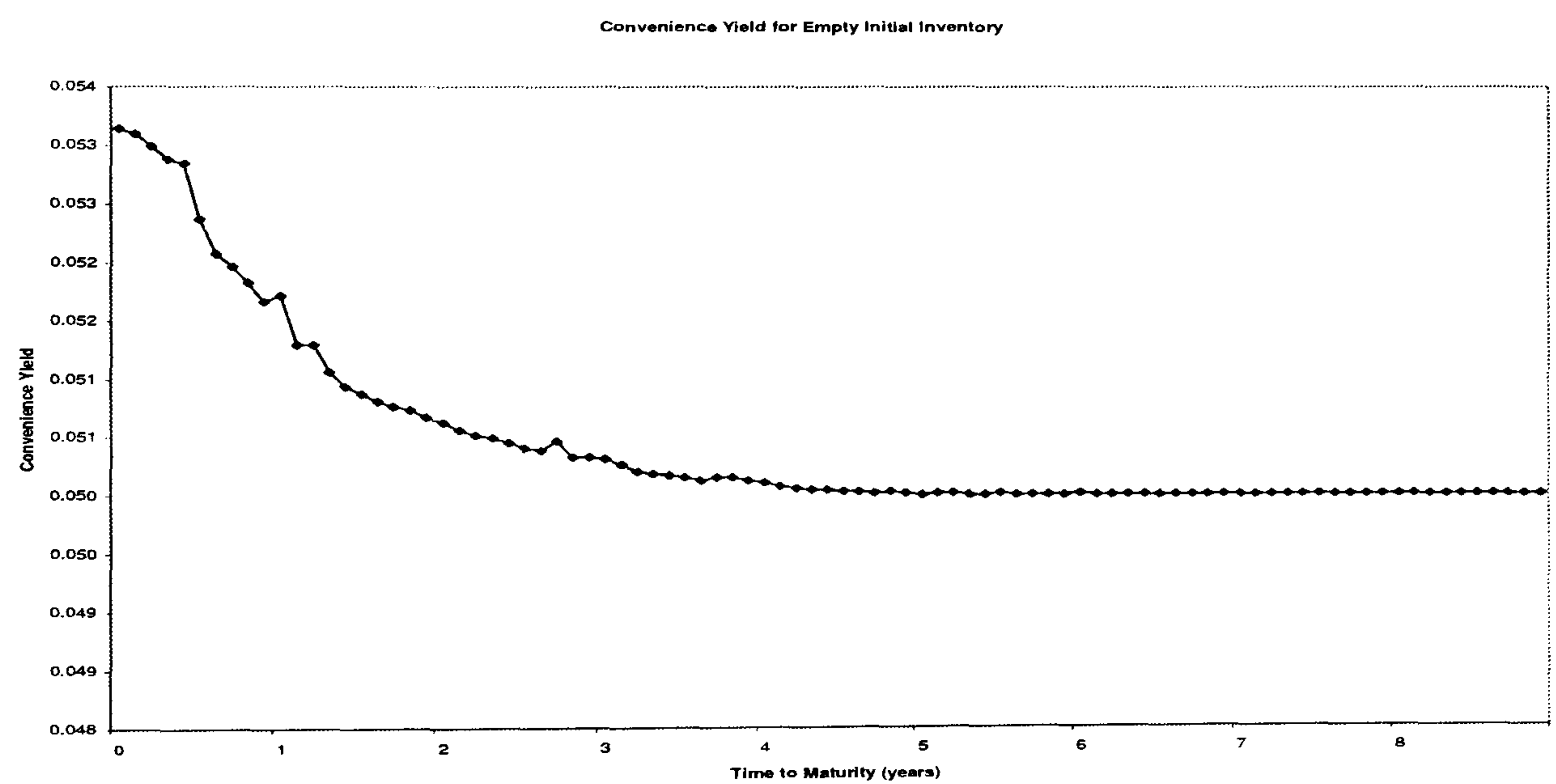


Figure 4.5: Monopolistic case: Evolution of the convenience yield curve when the initial supply rate is 4.5 and the initial inventory is empty.

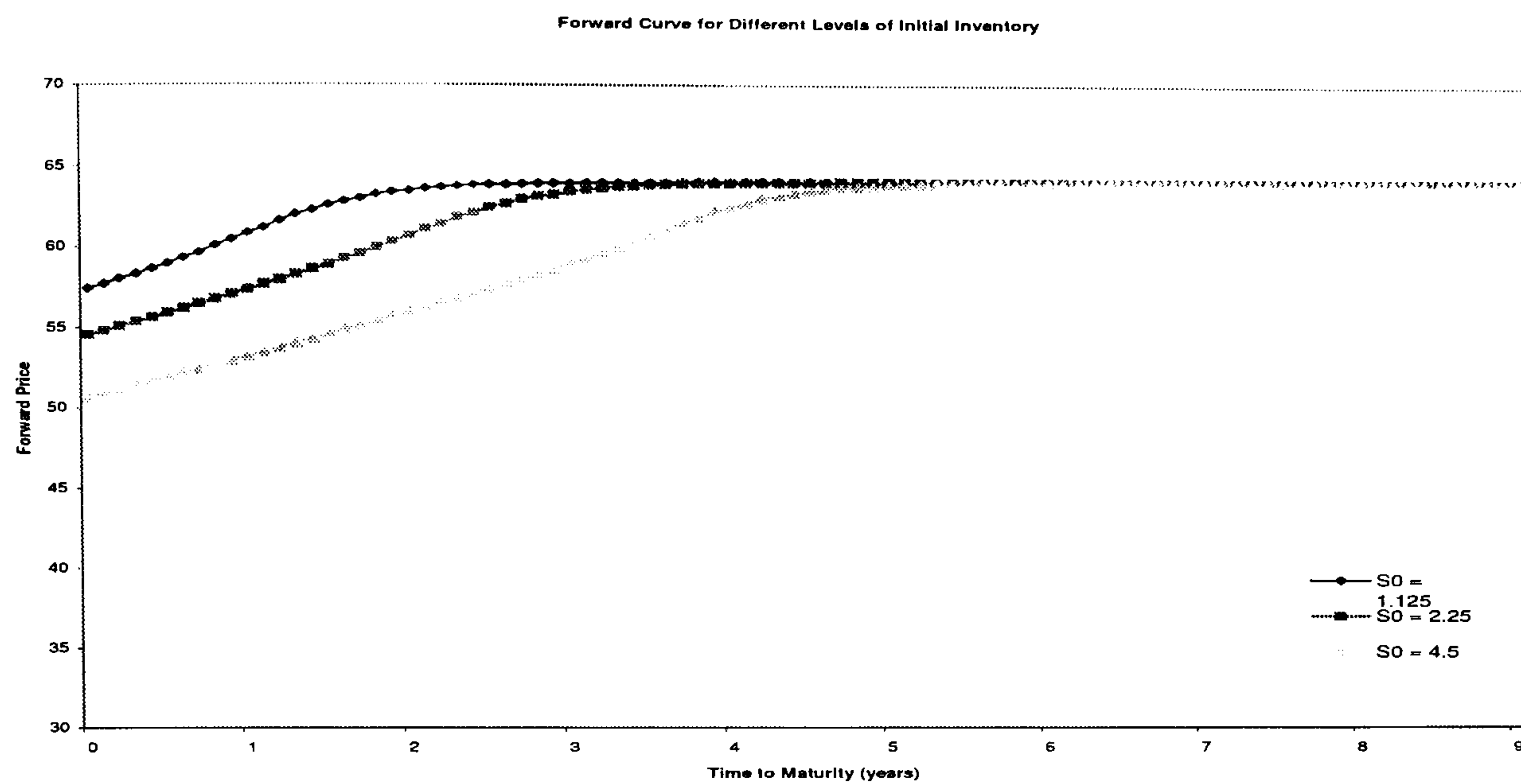


Figure 4.6: Competitive case: Evolution of the forward curve when the initial supply rate is 4.5 and the initial inventory is equal to 1.125, 2.25 and 4.5, respectively.

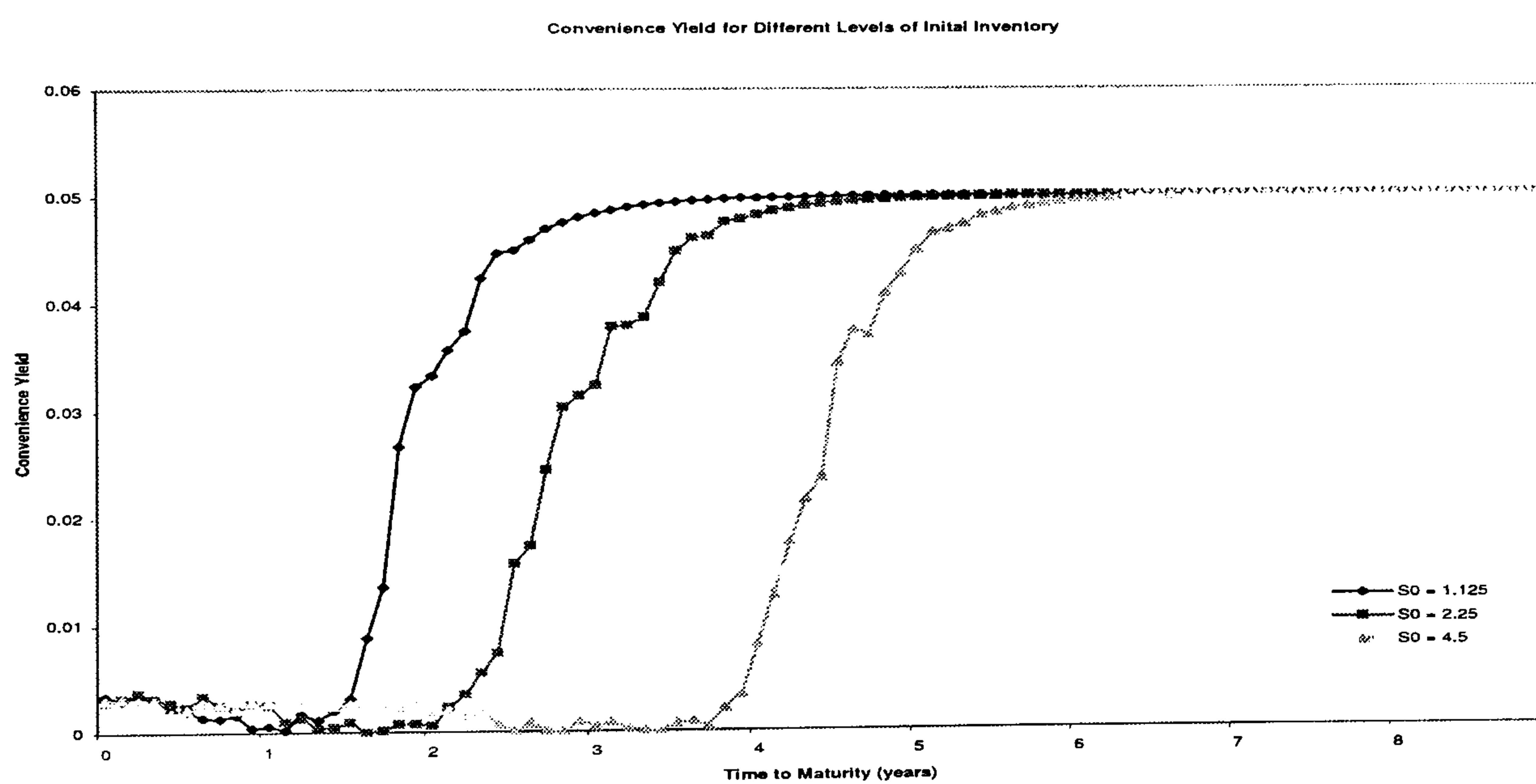


Figure 4.7: Competitive case: Evolution of the convenience yield curve when the initial supply rate is 4.5 and the initial inventory is equal to 1.125, 2.25 and 4.5, respectively.

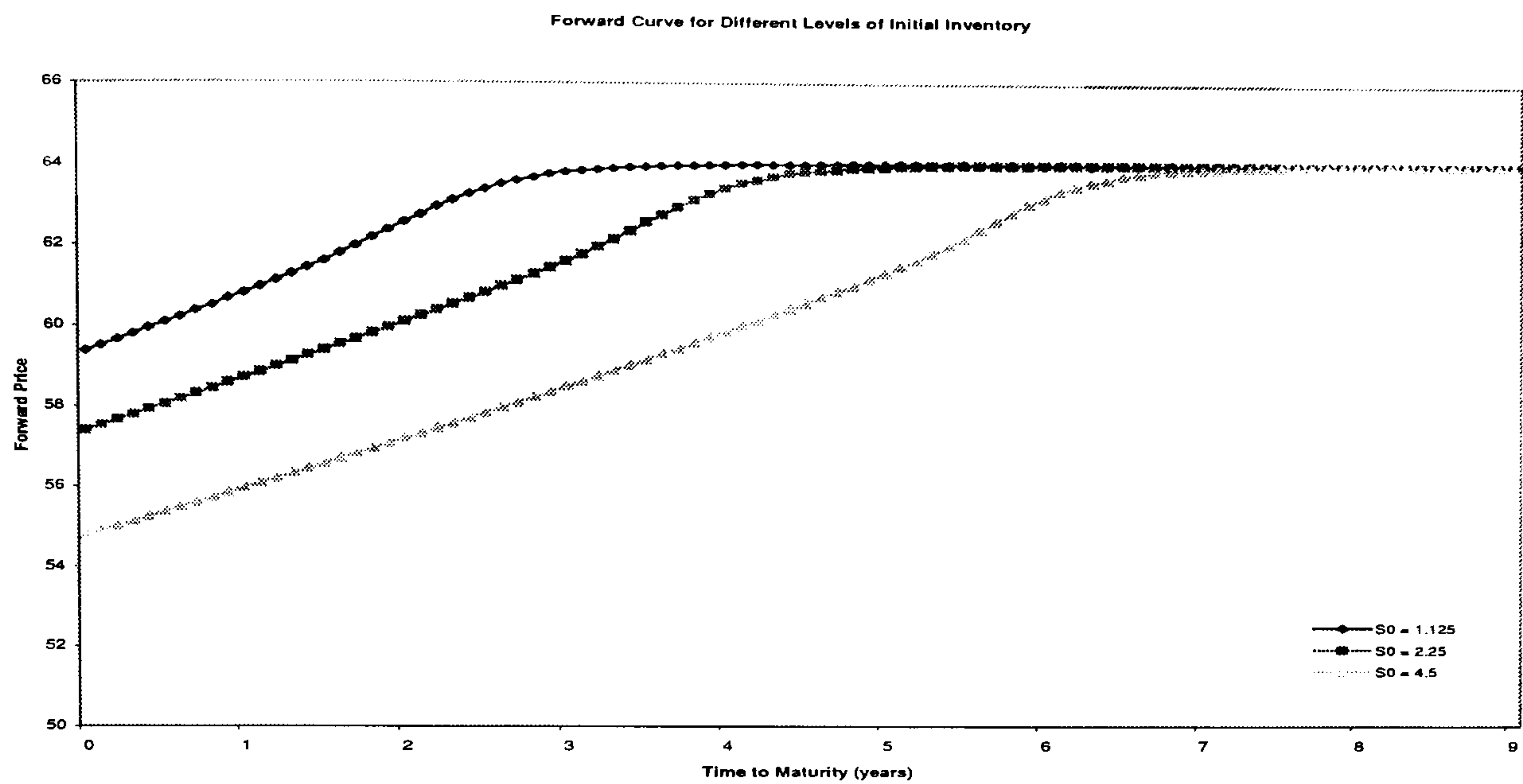


Figure 4.8: Monopolistic case: Evolution of the forward curve when the initial supply rate is 4.5 and the initial inventory is equal to 1.125, 2.25 and 4.5, respectively.

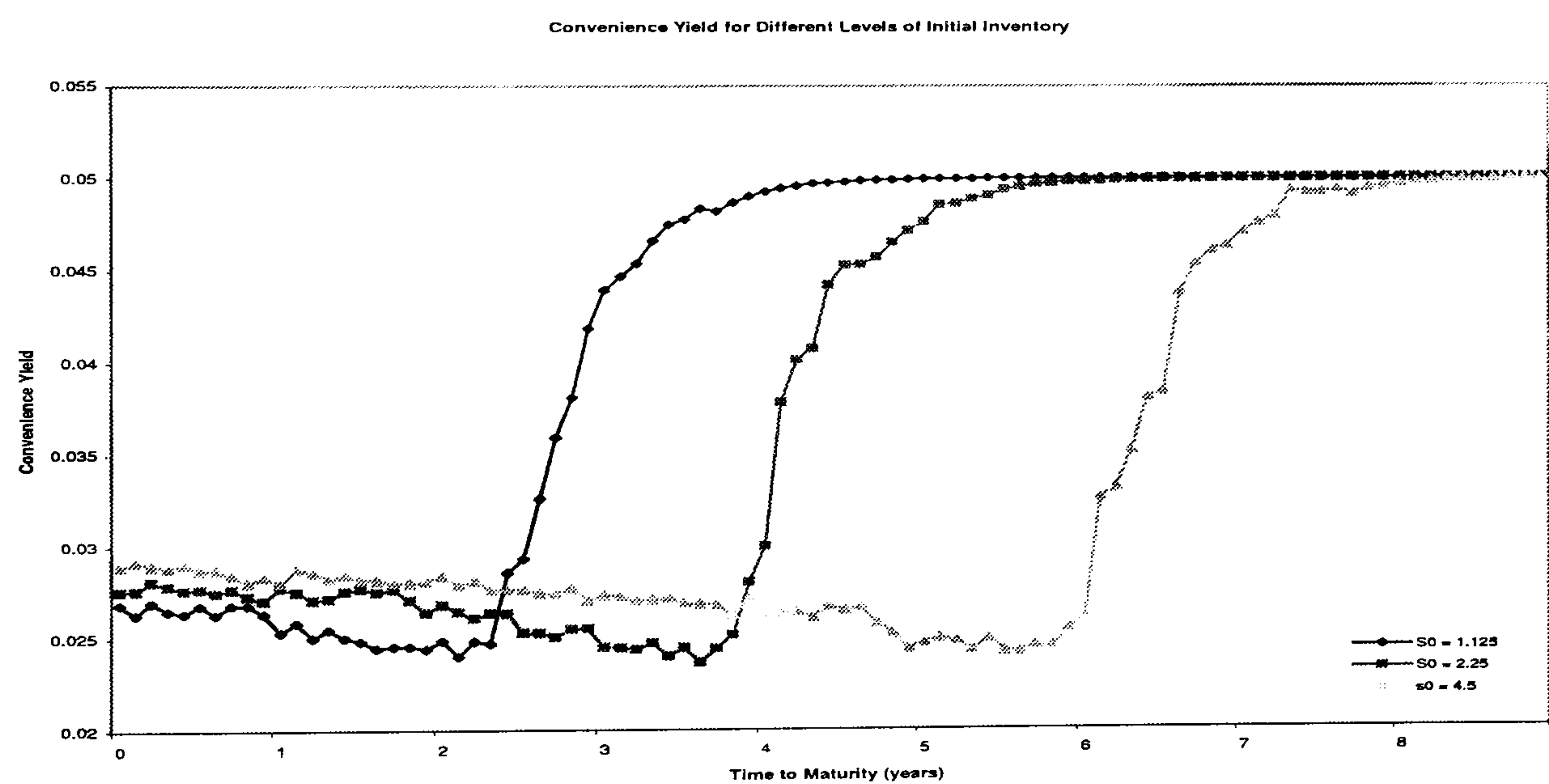


Figure 4.9: Monopolistic case: Evolution of the convenience yield curve when the initial supply rate is 4.5 and the initial inventory is equal to 1.125, 2.25 and 4.5, respectively.

4.4 Conclusion

In this chapter we implemented and analyzed the commodity forward curves which correspond to the storage structural model presented in the previous chapter. Thus, this chapter complements and concludes the analysis of the storage structural model developed in Chapter 3. In order to obtain the commodity forward curves we developed a sophisticated numerical procedure which is comparable to the one suggested by Hull and White (1993a) [55].

The forward curve analysis show that the commodity forward curves take two fundamental shapes depending on whether the initial commodity price is below or above the state-independent long-term forward price, $F_\infty = \bar{P}$. Specifically, if the commodity price is less than the long-run forward price, the curve will be in contango otherwise it will be in backwardation. These properties are consistent with the theory of storage and with the analysis provided by Routledge, Seppi and Spatt (2000) [78].

This chapter presents two main contributions to the literature. First, we developed a sophisticated numerical procedure which is inspired by the one suggested by Hull and White (1993a) [55] but significantly different. This procedure can be applied to any two-state dependent stochastic dynamic control problem for which there exists a steady state policy for the controllable variable. Second, this chapter provides a comprehensive analysis of the forward curve implied by a structural model. With the exception of Routledge, Seppi and Spatt (2000) [78], the existing literature in structural models for commodity prices limits the scope of analysis to the study of the spot price properties as a function of the state variables. Hence, these papers do not value forward curves or any other derivatives. Routledge, Seppi and Spatt (2000) [78] present a study of equilibrium forward

curves conditional on the initial inventory and demand levels⁹. Nevertheless, their analysis is limited to the case where the demand can only take two possible states - high and low - and it is very difficult to generalize to a more realistic number of demand states or to a more general Markov process. In contrast, the numerical procedure presented in this chapter is fairly general and our analysis can easily be generalized to any combination of initial values of the supply and inventory level. Furthermore, we also provide a comparison between the forward curves observed within the competitive and the monopolistic storage markets, which has not been yet considered in the literature.

This study suggests testable hypothesis concerning the empirical dynamics of commodity spot/futures prices and provided us with useful insights into the most desirable properties to incorporate into reduced form models which are presented in the following chapters.

⁹As mentioned before, the source of uncertainty in their model comes from the stochastic demand.

Chapter 5

A Contango Constrained Reduced Form Model

5.1 Introduction

In this chapter, we introduce a new reduced form model for continuously produced storable commodity prices. The model is developed under the risk neutral measure. We exploit this model to study the properties of the simplest mean reverting model possible which satisfies the arbitrage-free contango constraint. Although we classify it as a reduced form model since prices are generated by continuous time stochastic processes, it incorporates nonlinearity in the prices, similarly to the structural models. Namely, the commodity spot prices switches between two distinct processes depending on whether or not inventory is being held. Accordingly, there is a single critical point in our model. Below this critical point the stock is being stored, otherwise all the the inventory is sold.

In structural models (Samuelson, 1971 [80]; Williams and Wright, 1991 [89]; Deaton and Laroque, 1992; 1996 [24, 25]; Chambers and Bailey, 1996 [12]

and Routledge, Seppi and Spatt, 2000 [78]), this sort of non-linearity is introduced by the non-negativity constraint in the inventory and is the central equilibrium condition. In this model the nonlinearity arises from imposing a cash-and-carry arbitrage-free condition in the spot price process. Specifically, whenever the drift of the spot price exceeds the cost of carrying inventory (interest rate plus storage cost) the inventory is being held. Conversely, whenever the drift of the spot price is less than the cost of carry all the inventory is sold and the storage facility becomes empty. Whenever inventory is being held we assume that the spot price follows a Geometric Brownian Motion (GBM) with drift equal to the cost of carrying inventory. Otherwise, the price follows a mean reverting process of the Ornstein-Uhlenbeck (O-U) type¹. This model follows arbitrage-free arguments, since under the risk neutral measure since the commodity price process must have the drift less or equal to the cost of carry. This approach clearly diverges from the main stream of reduced form models in the literature (e.g. Gibson and Schwartz, 1990 [42]; Schwartz, 1997 [82]; Hilliard and Reis, 1998 [51]; Miltersen and Schwartz, 1998 [68]), where the spot price is defined by the same process, regardless inventory is being held or not.

The standard reduced form models approach considers that the spot price follows a log-normal diffusion process where the convenience yield is incorporated as an exogenous "dividend" yield. The mean reversion characteristic of the commodities spot and forward prices² is incorporated via the spot price itself (e.g. Schwartz, 1997 [82], single-factor model) or indirectly, via the convenience yield (Schwartz, 1997 [82] two-factor model; Gibson and Schwartz, 1990 [42]; Hilliard and Reis, 1998 [51] and Miltersen and Schwartz, 1998 [68]). These models are

¹This is the same as single-factor model described by Schwartz (1997) [82].

²Gibson and Schwartz (1990) [42], Brennan (1991) [9], Bessembinder *et al* (1995) [5] and Pilipovic (1997) [74]) found strong empirical evidence in favor of the mean reverting pattern of spot commodity prices.

widely used in practice mainly due to their high analytical tractability. However, these models have misspecification problems and therefore may generate errors in pricing commodity contingent claims, particularly during periods of market instability (see Pirrong, 1998 [75] and Clewlow and Strickland, 2000 [16]). We believe that these misspecifications partly arise from an inappropriate definition of the convenience yield. One-factor models (Brennan and Schwartz, 1985 [10] and Schwartz, 1997 [82]) consider a constant convenience yield and ignore the fact that convenience yield strongly depends on the supply, demand and inventory conditions in the market. The two- and three-factor models (Gibson and Schwartz, 1990 [42]; Schwartz, 1997 [82]; Hilliard and Reis, 1998 [51] and Miltersen and Schwartz, 1998 [68]) typically consider that the spot price follows a standard GBM process and the convenience yield is an exogenous factor that follows a mean-reverting O-U stochastic process, which has a constant correlation with the spot price³. The specification of the convenience yield as an exogenous stochastic process and its inclusion in the spot price process as a "dividend" yield seems ilusive. In fact, the convenience yield is not an observable variable but it is derived from the relationship between spot and forward commodity prices. Consequently, this approach might generate inconsistencies between the spot and forward prices. Additionally, modelling the convenience yield as a O-U stochastic process might generate negative values and therefore create violations of the cash-and-carry arbitrage-free condition.

The model presented in this chapter essentially focuses on replicating the mean-reverting characteristic of commodity spot prices likewise current models and

³The three factor models only differ from the two factor models by incorporating a stochastic interest rates. Nevertheless, the inclusion of stochastic interest rates in the commodity price models does not have a significant impact in pricing commodity contingent claims (see Clewlow and Strickland, 2000 [16]). Therefore there is not advantage in losing tractability by incorporating stochastic interest rates.

on ensuring absence of arbitrage by restricting the spot price drift. We are particularly interested in analyzing the spot price and the forward curve properties implied by this model. We illustrate and analyze these properties by applying trinomial tree techniques to compute numerical examples of spot price sample paths and the corresponding forward curves and spot price distribution. For comparison, we also provide equivalent numerical examples for the single-factor model described by Schwartz (1997) [82]. This model defines the spot price process as a single mean reverting process of the O-U type and corresponds to the unconstrained version of the spot price process in our model. This comparison has two purposes. The first is to compare the properties of our model with the single-factor model developed by Schwartz. The second is to understand how the introduction of the possibility of storage in our model changes the commodity forward curve and the spot price distribution generated by a mean reverting stochastic process.

The remaining of this chapter is organized as follows. Section 5.2 defines the model. Section 5.3 describes the numerical implementation. Section 5.4 presents and analyzes the numerical results. Section 5.5 concludes.

5.2 Model Definition

The model is specified under the risk-neutral measure. We assume that the risk-free interest rate, r , is constant and therefore forward and futures prices are equivalent. Thereafter, we may refer to one or another without distinction. We also assume that the storage cost is given by $c \times p_t$ per unit stored, where $0 < c < 1$ represents a proportion of the spot price.

The commodity spot price process switches between two distinct stochastic processes, depending on whether or not inventory is being held. In the absence of storage, the spot price follows a mean reverting process of the O-U type. Whenever

the drift in the spot price, p_t , exceeds the cost of carrying inventory (interest rate, r , plus storage cost, c) the stock holders hold stock. Whenever inventory is being held, the commodity price has the drift equal to $(r + c)$. Therefore, we assume that the spot price process follows a standard GBM whenever the stock is being held. This rule leads to the existence of a single critical price, p^* , in our model. Accordingly, the inventory holders buy stock as soon as p_t falls below p^* . On the other hand, as soon as p_t rises above p^* , all the inventory is sold. In the latter case, the spot price switches back to the mean-reverting stochastic process.

Let p_t be the commodity spot price. The stochastic process for the spot price switches between the two following components:

$$dp_t = \alpha(m - \ln p_t)p_t dt + \sigma p_t dB_t, \quad \text{if } p_t \geq p^* \quad (5.1)$$

$$dp_t = (r + c)p_t dt + \sigma p_t dB_t, \quad \text{otherwise,} \quad (5.2)$$

where:

- p_t is the spot price;
- m is a constant;
- α is the speed of mean reversion, a constant;
- σ is the volatility of the spot price, a constant;
- B_t is a standard Wiener process;
- r is the constant risk-free rate;
- c is the storage cost, which is a constant proportion of the spot price;
- p^* is the critical spot price.

Following the description above, the critical value spot price, p^* , is given by:

$$(r + c) = \alpha(m - \ln(p^*)), \quad \text{that is,} \quad (5.3)$$

$$\ln(p^*) = m - \frac{(r + c)}{\alpha} \quad (5.4)$$

Equation (5.1) represents an alternative to the standard geometric O-U process adopted by Dixit and Pindyck (1994) [28], Metcalf and Hasset (1995) [67] and Epstein et al. (1998) [35] in the context of real options. This alternative eases the numerical implementation of the O-U process. Possibly also due to this advantage, Schwartz (1997) [82] also uses this format to represent the single-factor model of a mean-reverting commodity price. Accordingly, Schwartz single-factor model corresponds to the unconstrained version of our model, that is, in the absence of the spot price drift constraint.

Defining $x_t = \ln p_t$ and applying Ito's lemma⁴, the log price follows the O-U stochastic process:

$$dx_t = \alpha(\bar{x} - x_t)dt + \sigma dB_t \quad \text{if} \quad x_t \geq x^* \quad (5.5)$$

$$dx_t = \left((r + c) - \frac{1}{2}\sigma^2 \right) dt + \sigma dB_t, \quad \text{otherwise} \quad (5.6)$$

where $x^* = \ln(p^*)$

The relation between \bar{x} and m is given by:

$$\bar{x} = m - \frac{\sigma^2}{2\alpha} \quad (5.7)$$

⁴This derivation is described in appendix E.

The log price, x_t , corresponding to the O-U process given by equation (5.5) is normally distributed with mean and variance given by:

$$E[x_T|x_t] = \bar{x} + (x_t - \bar{x})e^{-\alpha(T-t)} \quad (5.8)$$

$$V[x_T|x_t] = \frac{\sigma^2}{2\alpha} (1 - e^{-2\alpha(T-t)}) \quad (5.9)$$

The branch of x_t corresponding to the GBM given by equation (5.6) is normally distributed with mean and variance given by:

$$E[x_T|x_t] = x_t + \left((r + c) - \frac{1}{2}\sigma^2 \right) (T - t) \quad (5.10)$$

$$Var[x_T|x_t] = \sigma^2(T - t) \quad (5.11)$$

Note, however, that the commodity price keeps switching between the two processes above and therefore none of the equations (5.8), (5.9), (5.10) and (5.11) describe the conditional moments of the resulting price distribution. However, the knowledge of this moments will support the numerical implementation of the commodity price lattice.

Under the risk-neutral measure, the futures price at time t for delivery at some future time $T > t$ is given by the expected spot price at time T conditional on the information available at t , that is:

$$F_{t,T} = E_t[p_T], \quad (5.12)$$

where E_t denotes the conditional expectation under the risk neutral measure given the information at time t . We apply this relationship to compute the forward curve for the commodity prices in the numerical implementation as described in the next section.

As in Chapter 4, we compute the convenience yield from the relationship between the futures and the spot price of a commodity when the interest rate and the convenience yield are deterministic. Since the storage costs are expressed as a proportion c of the spot price, the convenience yield, δ is defined so that:

$$F_{t+\Delta t} = p_t e^{((r+c)-\delta)\Delta t} \quad (5.13)$$

Based on this relationship, we calculate the annualized convenience yield for the time interval between t and $t + \Delta t$ by using pairs of adjacent maturities futures contracts according to the following formula:

$$\delta_{t,t+\Delta t} = (r + c) - \frac{1}{\Delta t} \ln \left(\frac{F_{t+\Delta t}}{F_t} \right) \quad (5.14)$$

Accordingly, the curve is in backwardation if the convenience yield, $\delta_{t,t+\Delta t}$ is greater than $(r+c)$; the curve is in contango if the convenience yield is less than $(r+c)$, that is, the forward curve increases with time to maturity. A negative convenience yield would imply the violation of the standard arbitrage-free condition for commodity prices:

$$F_{t+\Delta t} \leq p_t e^{(r+c)\Delta t}. \quad (5.15)$$

5.3 Numerical Implementation

We apply trinomial tree techniques to illustrate numerical examples of the forward curve and to analyze the properties implied by this model. In particular, we aim to study the properties of the spot price distribution, the forward curve and the corresponding convenience yield. In order to understand the dynamics generated by imposing a constraint on the drift of the spot prices in our model, we also compute a numerical examples of the unconstrained version of our model. As mentioned above, the unconstrained version of our model is the same as the Schwartz (1997) [82] single-factor model.

Excess kurtosis and right skewness are two fundamental properties typically observed in storable commodity price distributions. In other words, commodity prices are characterized by long periods of stagnant prices interrupted by sharp upward prices. This asymmetry is a consequence of the inventory non-negativity constraint. Inventory can always be added to keep current spot prices from being too low. In other words, the existence of storage "cuts out" the left tail of the price distribution. However, stockholders are unable to respond to sudden demand/supply imbalances that lead to sudden upward rises in commodity prices. Hence, we are particularly interested in verifying if our model generates right skewness and excess kurtosis in the commodity prices distribution.

We compute a single tree for the logarithm of the commodity prices paths, $x_t = \ln(p_t)$, which results from the combination of equations (5.5) and (5.6). That is, x_t follows a O-U process described by equation (5.5) if $x_t \geq x^*$ and follows the process given by equation (5.6) otherwise. We describe separately the local probability structure of the tree corresponding to each of the processes and then explain how to combine both procedures in order to obtain the final lattice for the model.

5.3.1 Lattice Description

A more detailed description of the trinomial lattice method for mean reverting stochastic process is presented in the appendix C. Nevertheless, we repeat here the main steps of the procedure. The method used to implement a trinomial tree to represent the standard O-U process given by equation (5.5) is based on the technique described in Hull and White (1993b, 1994) [56, 57] and later revised by Clewlow and Strickland (1998) [14] and Clewlow and Strickland (2000) [16], which was originally designed to implement short interest rates that follow a mean-reverting arithmetic stochastic process.

The trinomial tree is constructed by using time steps of length Δt and x -steps of length Δx . At the end of each time step, x takes the value $x_0 + j\Delta x$, where j can be either positive or negative and x_0 is the initial value. (i, j) is defined as the node for which $t = i\Delta t$ and $x = j\Delta x$. As described in appendix C, the trinomial branching process can take three different forms. It can take a normal branching process where we can move up by Δx , stay the same and move down by Δx ; a branching process when $x_{i,j}$ is currently low and $x_{i,j}$ can stay the same, move up by Δx and move up by $2\Delta x$. Finally when $x_{i,j}$ is currently high, the price path can stay the same, move down by Δx and move down by $2\Delta x$ ⁵. In other words, the three nodes emanating from node (i, j) are $(i + 1, k + 1)$ - the "upper" node, $(i + 1, k)$ - the "middle" node and $(i + 1, k - 1)$ - the "lower" node. The value of k is chosen so that $x_{i+1,k}$ is as close as possible to the expected value of x , which by definition is given by $x_{i,j} + \mu_{i,j}$, where $\mu_{i,j} = \alpha(\bar{x} - (x_0 + j\Delta x))\Delta t$. For the normal branching process $k = j$, when $x_{i,j}$ is currently low $k = j + 1$ and when $x_{i,j}$ is currently high $k = j - 1$, respectively.

In order to obey stability and convergence conditions Hull and White (1990a)

⁵See Figure C.1 in appendix C.

[53] suggest that a good relationship between Δt and the space step Δx is:

$$\Delta x = \sigma \sqrt{3\Delta t} \quad (5.16)$$

Define (i, j) as the node for which $t = i\Delta t$ and $x = j\Delta x$. Define $pr_{i,j}^u$, $pr_{i,j}^m$ and $pr_{i,j}^d$ as the probabilities of the highest, middle and lowest branches emanating from node (i, j) . The probabilities are chosen to match the expected change and variance in x over the next interval Δt . The probabilities must also sum to unity. Accordingly, the resulting probabilities are given by:

$$\begin{aligned} pr_{i,j}^u &= \frac{\sigma^2 \Delta t + \eta^2}{2\Delta x^2} + \frac{\eta}{2\Delta x} \\ pr_{i,j}^m &= 1 - \frac{\sigma^2 \Delta t + \eta^2}{\Delta x^2} \\ pr_{i,j}^d &= \frac{\sigma^2 \Delta t + \eta^2}{2\Delta x^2} - \frac{\eta}{2\Delta x} \end{aligned} \quad (5.17)$$

where $\eta = \mu_{i,j} + (j - k)\Delta x$ and $k = j - 1, j$ and $j + 1$, depending on the type of branching, as described above and $\mu_{i,j} = E[\Delta x | (i, j)] = \alpha(\bar{x} - x_{i,j})\Delta t$ is the conditional expectation of the discretized x_t process at node (i, j) and $\sigma_{i,j}^2 \Delta t + (\mu_{i,j} \Delta t)^2 = E[\Delta x^2 | (i, j)]$, where $\sigma_{i,j} = \sigma$ is from equation (5.5). Provided that Δx is within the range $\sigma\sqrt{3\Delta t}/2$ to $2\sigma\sqrt{\Delta t}$, the probabilities are always between 0 and 1 (Hull and White, 1993b [56]).

The implementation of a trinomial lattice for the GBM is simpler than it is in the case of a mean reversion stochastic process as described above. The procedure is standard (see, e.g. Clewlow and Strickland, 1998 [14]). Now, there are no branching decisions to be made, that is, we always have a "normal" branching process.

As before, we work in terms of $x_t = \ln(p_t)$. The trinomial tree is constructed by using time steps of length Δt and x -steps of length Δx . Again, at the end of each time-step, x takes the value $x_0 + j\Delta x$, where j can be either positive

or negative and x_0 is the initial value. Each node of the tree is represented by (i, j) , for which $t = i\Delta t$ and $x = j\Delta x$. At each node, $x_{i,j}$ can go up by Δx , stay the same or go down by Δx , with probabilities p^u , p^m and p^d , respectively. As before, we choose $\Delta x = \sigma\sqrt{3\Delta t}$. The probabilities are obtained by matching the mean and variance over the time interval Δt and requiring that the probabilities sum to one:

$$\begin{aligned} p_u + p_m + p_d &= 1 \\ (\Delta x)p_u + 0p_m + (-\Delta x)p_d &= \left((r + c) - \frac{1}{2}\sigma^2 \right) \Delta t \\ (\Delta x^2)p_u + 0p_m + (\Delta x^2)p_d &= \sigma^2 \Delta t + \left((r + c) - \frac{1}{2}\sigma^2 \right)^2 \Delta t^2 \end{aligned} \quad (5.18)$$

where $\left((r + c) - \frac{1}{2}\sigma^2 \right) \Delta t = E[\Delta x|(i, j)] = \mu$ and $\sigma^2 \Delta t + \left((r + c) - \frac{1}{2}\sigma^2 \right)^2 \Delta t^2 = \sigma^2 \Delta t + \mu^2 \Delta t^2 = E[\Delta x^2|(i, j)]$. Solving these equations we obtain:

$$\begin{aligned} p_u &= \frac{1}{2} \left(\frac{\sigma^2 \Delta t + \mu^2}{\Delta x^2} + \frac{\mu}{\Delta x} \right) \\ p_m &= 1 - \frac{\sigma^2 \Delta t + \mu^2}{\Delta x^2} \\ p_d &= \frac{1}{2} \left(\frac{\sigma^2 \Delta t + \mu^2}{\Delta x^2} - \frac{\mu}{\Delta x} \right) \end{aligned} \quad (5.19)$$

The value of the commodity price relative to the initial commodity price at node (i, j) , which corresponds to the i^{th} time step and level j in the tree is $p_{i,j} = p_0 \exp\{j\Delta x\}$.

The complete tree is constructed according to space-time description above, that is, using time-steps Δt and x -steps of length $\Delta x = \sigma\sqrt{3\Delta t}$. Each node is represented by (i, j) for $i = 0, \dots, N$ and $j = j_{min}, \dots, j_{max}$, where N is the total

number of time-steps and j_{min} and j_{max} are the minimum and maximum levels in the tree at each time step i . At every node, we test whether the commodity spot price, p_t , is greater than the threshold price, p^* , and choose the probabilities $p_{i,j}^u$, $p_{i,j}^m$ and $p_{i,j}^d$ accordingly. That is, the local probabilities are defined by (5.17) if $p_{i,j} > p^*$, which means that the price follows the stochastic process described by (5.5); otherwise the local probabilities are given by (5.19). Note that the branching decision between (a), (b) and (c) in Figure C.1 in appendix C is only necessary to be considered when p_t follows the mean-reverting process; otherwise, we have the "normal" branching process as described above.

5.4 Numerical Results

5.4.1 Results from the Lattice Model

In this section we illustrate and analyze examples of forward curves generated by both our model and by the mean-reverting single-factor model described by Schwartz (1997) [82]. As mentioned before, this model represents the unconstrained version of our model. This enables us to study the effect of the introduction of storage through the imposition of an arbitrage-free constraint in our model. The lattices are implemented for a period of five years, that is, from $t = 0$ to $T = 5$ using the parameter values displayed in Table 5.1 below.

α	σ	\bar{x}	r	c
3	0.2	$\ln 45$	0.05	0.1

Table 5.1: Values of the parameters used in the lattice computation.

Figure 5.2 illustrates a set of forward curves generated by our model, which corresponds to five different spot price values at time $t = 0$, that is,

$p_0 = 25, 35, 45, 55$ and 65 . This range of values are representative for the spot prices that might be generated by this model for the parameters shown in Table 5.1. Figure 5.1 illustrates the forward curves produced by the unconstrained single-factor Schwartz model. These curves are computed for the same range of initial spot prices and the same parameters values.

Figures 5.2 and 5.1 show that, for high spot prices at time $t = 0$, the curves are in backwardation; conversely, for low spot prices at $t = 0$, the curves are in contango. In either case, the curves eventually move towards a long-run state independent forward price, F_∞ . For the unconstrained version of our model, the long-run mean is equal 44.85, which is the long-run mean of the O-U process for the spot price⁶. For our model, the long-run mean of the spot price is slightly lower and approximately equal to 42.3. However, when $p_0 = 25$ the forward curve implied by our model does not reach the long run steady price, \bar{p} , for the five year period considered. We will discuss this issue later below. The length of time each curve remains in contango/backwardation is directly proportional to the distance between the spot price at time zero and the long-run equilibrium price, \bar{p} .

Figures 5.4 and 5.3 illustrate the corresponding convenience yield curves. If the initial spot price, p_0 , is significantly above the long-run mean, the corresponding convenience yield is also high. High spot prices signal tight demand and supply conditions in the market, which imply the possibility of a stockout. This in turn leads to high convenience yield values since there is a high benefit from holding inventory. As a result we observe backwardation. On the other hand, if the spot prices are low because there is low-demand/high supply in the market, the stockholder builds inventory in the expectation of a rise in the spot prices. These circumstances imply a low convenience yield and therefore we observe contango.

⁶In the long-run, x_t follows a normal distribution with $E[x_t] = m - \sigma^2/2\alpha = 3.80$. Since $p_t = \exp(x_t)$, the long-run mean for the spot price is $\bar{p} = E[p_t] = \exp(m - \sigma^2/4\alpha) = 44.85$.

Both these conditions are only temporary because the supply/demand conditions in the market tend to adjust and the forward curves move towards an equilibrium long-run forward price, say $F_\infty = \bar{p}$.

The main differences between the forward curves implied by the unconstrained single-factor model by Schwartz (1997) [82] and the constrained model presented here are observed when the market is in contango, while the forward curves are similar in backwardation. The degree of contango measured by the slope of the convenience yield is much greater in the unconstrained model, whereby we observe negative values in the convenience yield. This implies that the arbitrage-free condition given by equation (5.15) is violated. We also observe that the contango and the backwardation phenomenon observed in the single-factor model is symmetric, whereas it is asymmetric in our model. This is an immediate consequence of the nature of the arbitrage-free condition that restricts the spot price drift when stock is being held, which only affects the contango relationship in the forward curve.

We would expect that the futures return is equal to the cost of carrying inventory when the curve is in contango and it is equal to zero in the long run since the forward price becomes equal to a steady-state constant, $F_\infty = \bar{p}$. This property is consistent with the forward curve analysis presented by Routledge, Seppi and Spatt (2000) and in Chapter 4. However, the convenience yield implied by the model presented here is initially equal to zero and starts to rise slowly afterwards until it becomes equal to the cost of carry. Moreover, if the initial spot price is very low compared with \bar{p} (for example when $p_0 = 25$), the convenience yield and therefore the forward curve do not reach the steady-state for the period of five years considered. In particular, the futures returns given by the slope of the forward curve, do not cover the cost of holding inventory.

Figures 5.5 and 5.6 illustrate the probability density functions for the spot prices sample paths at final time $T = 5$ when $p_0 = 45$ (without loss of generality) generated by our model and by the unconstrained mean reverting model, respectively. Table 5.2 shows corresponding sample moments. As expected, the unconstrained model presents a perfectly centred distribution with a Gaussian kurtosis since the log spot prices generated by this model have a steady-state normal distribution in the long-run. The introduction of the constraint in the spot prices drift skews the spot price distribution to the left and adds a significant amount of kurtosis. The latter property is desirable commodity prices typically exhibit high kurtosis. However, the left skewness is not a property that we usually observe in commodity markets. On the contrary, commodity prices are generally right skewed, since we frequently observe upward (not downward) price spikes in commodity markets. As mentioned previously, this behaviour results from the non-negativity constraint in the inventory.

It is interesting to compare the results implied by this model with the results implied by the competitive structural model developed in Chapters 3 and 4. The shape of the forward and convenience yield curves implied by each of these models is similar when the market is in backwardation. However, the shape of these curves is significantly different when the market is in contango. The structural model implies that the convenience yield is either equal to zero when the market is in contango and or it is equal to the cost of carry in the steady-state. This implies that the futures return is equal to the cost of carrying inventory when the curve is in contango and it is equal to zero in the long run since the forward price becomes equal to a steady-state constant, $F_\infty = \bar{p}$. This property is consistent with the forward curve analysis presented by Routledge, Seppi and Spatt (2000) [78]. In the model presented in this chapter, the convenience yield is initially equal to zero

and starts to rise slowly afterwards until it becomes equal to the cost of carry. In particular, if the initial spot price is very low compared with \bar{p} (for example $p_0 = 25$), the convenience yield and therefore the forward curve do not reach the steady-state. In particular, this implies that the futures returns, given by the slope of the forward curve, do not cover the cost of holding inventory. This diverges from the forward curve properties implied by the structural model developed in Chapter 3 and from the structural models in the literature. Moreover, this behaviour is not realistic, since it implies that the stockholders do not cover the cost of carry when the forward price is in contango.

Summarizing the results in this section, we conclude that our model generates a rich set of forward curves without violating arbitrage-free conditions. The general properties of the forward curves are consistent with the theory of storage. In particular, low initial spot prices generate contango and high initial spot prices generate backwardation. In addition, the degree of backwardation (contango) is increasing (decreasing) with the initial spot prices. In the long run, however, the forward move towards a long-run value, $F_\infty = \bar{p}$, which is independent of the initial spot price. We ensure that the convenience yield is always positive and therefore this model does not violate arbitrage-free conditions. Additionally, our model implies that the spot price distribution presents excess kurtosis, which is a property observed in most commodity markets. From this perspective, our model is more appropriate than the standard ones such as Gibson and Schwartz (1990) [42], Schwartz (1997) [82], Miltersen and Schwartz (1998) [68] and Hilliard and Reis (1998) [51]). Nevertheless, this model also presents some misspecifications both in the statistical properties of the spot price distribution and the shape of the forward curve when the market is in contango. In particular, the spot price distribution is left skewed instead of being right skewed as desirable; the shape of

the forward curve in contango is not realistic since stockholders are not able to cover the cost of carrying inventory when the forward curve is upward sloping.

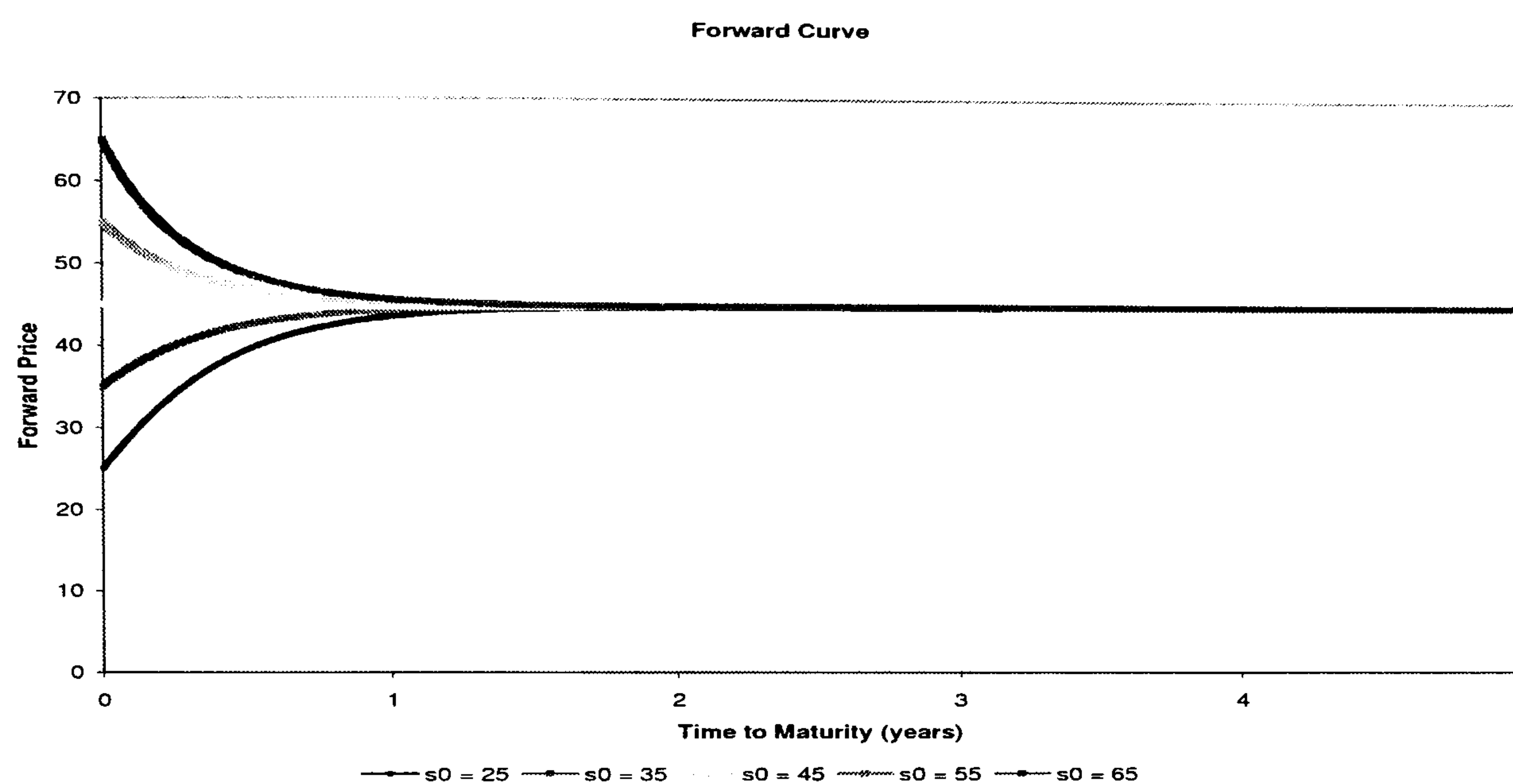


Figure 5.1: Forward curves generated by the one-factor mean-reverting model at different spot price levels for a 5-year period.

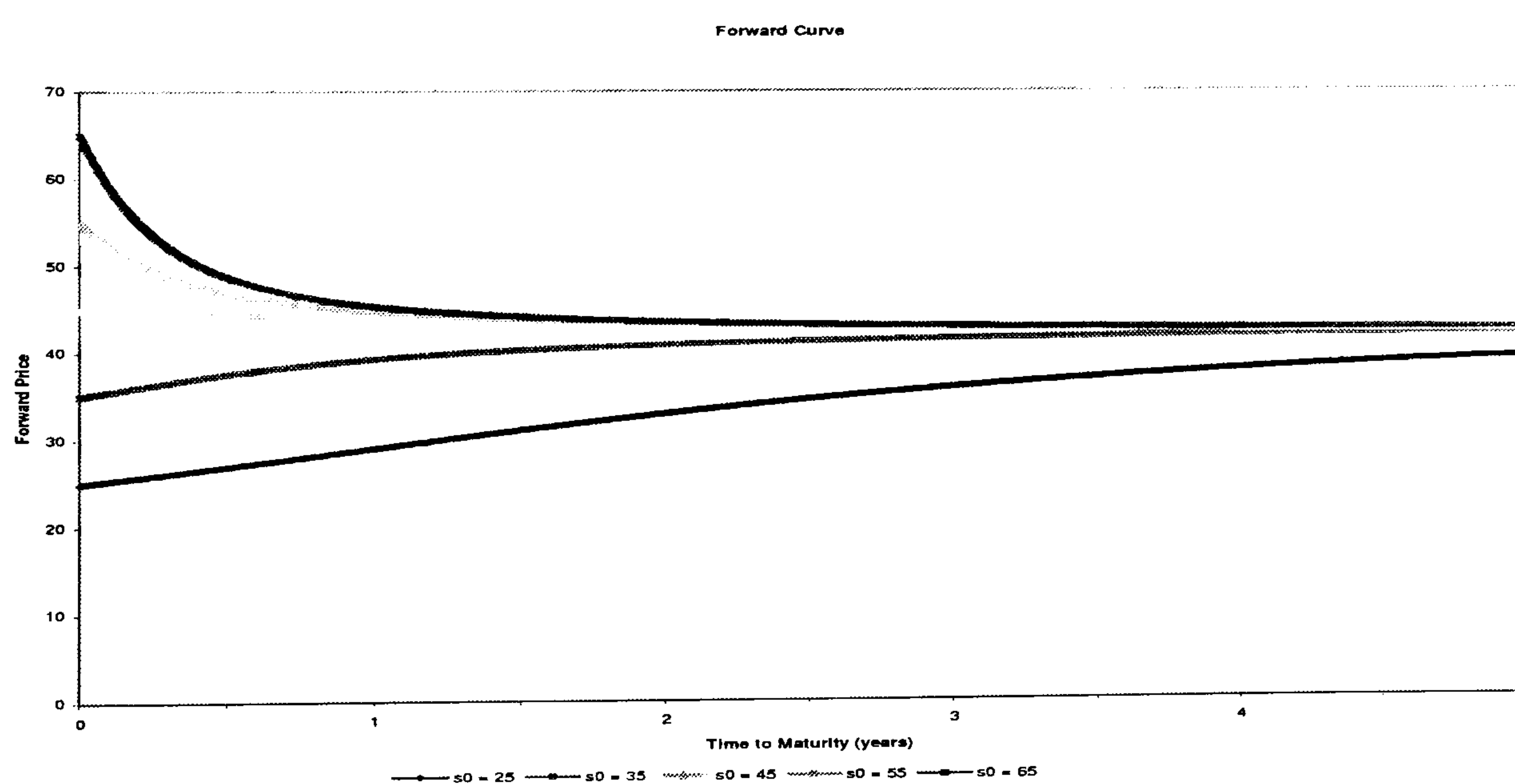


Figure 5.2: Forward curves generated by our model at different spot price levels for a 5-year period.

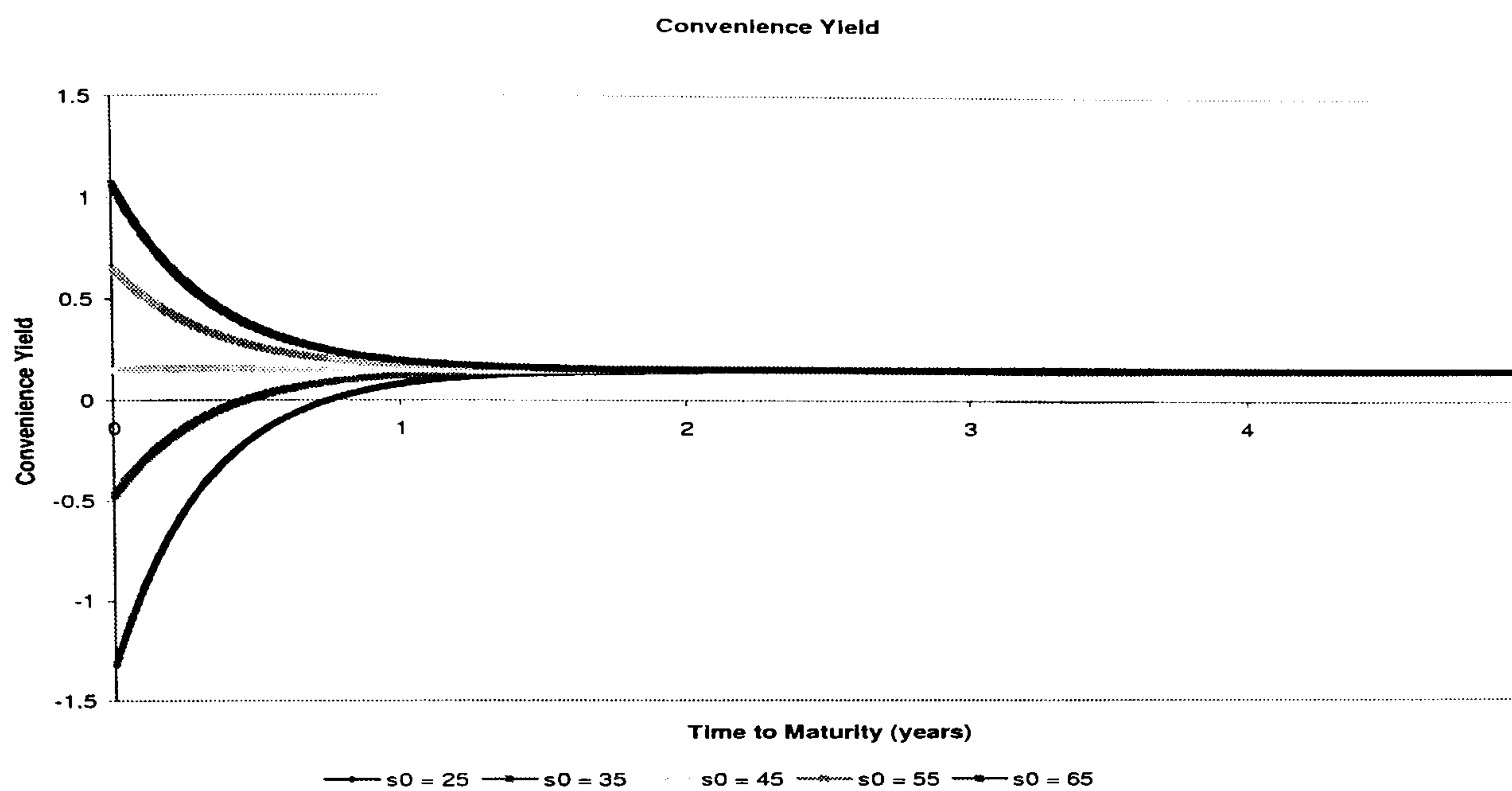


Figure 5.3: Convenience yield term structures generated by the one-factor mean-reverting model at different spot price levels for a 5-year period.

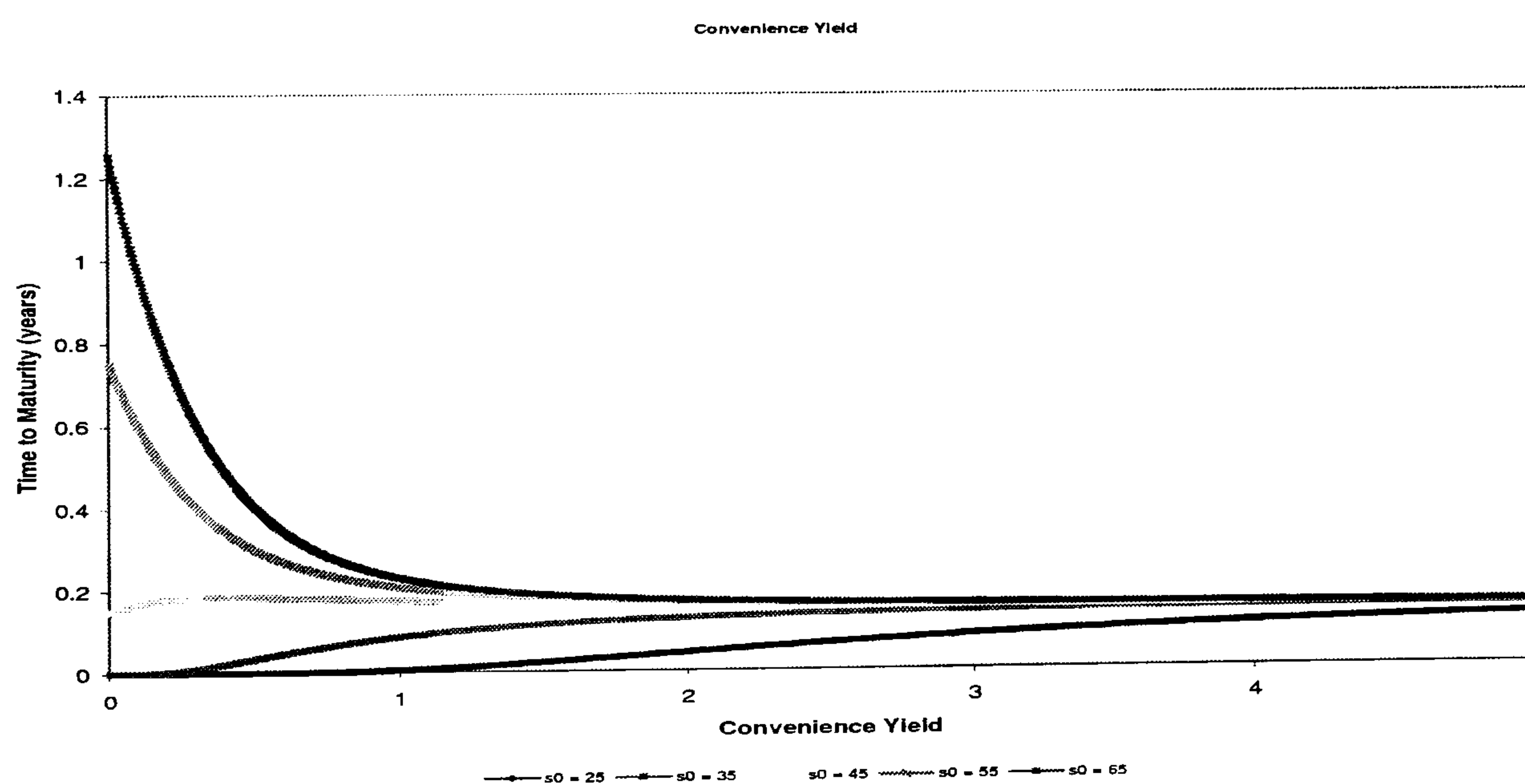


Figure 5.4: Convenience yield term structures generated by our model at different spot price levels for a 5-year period.

	<i>Mean</i>	<i>St.Dev.</i>	<i>Skewness</i>	<i>Kurtosis</i>
Our Model	3.73	0.15	-1.35	6.07
Unconstrained Model	3.80	0.08	0.00	3.00

Table 5.2: Sample moments at final $T = 5$ for the log price sample paths x_T .

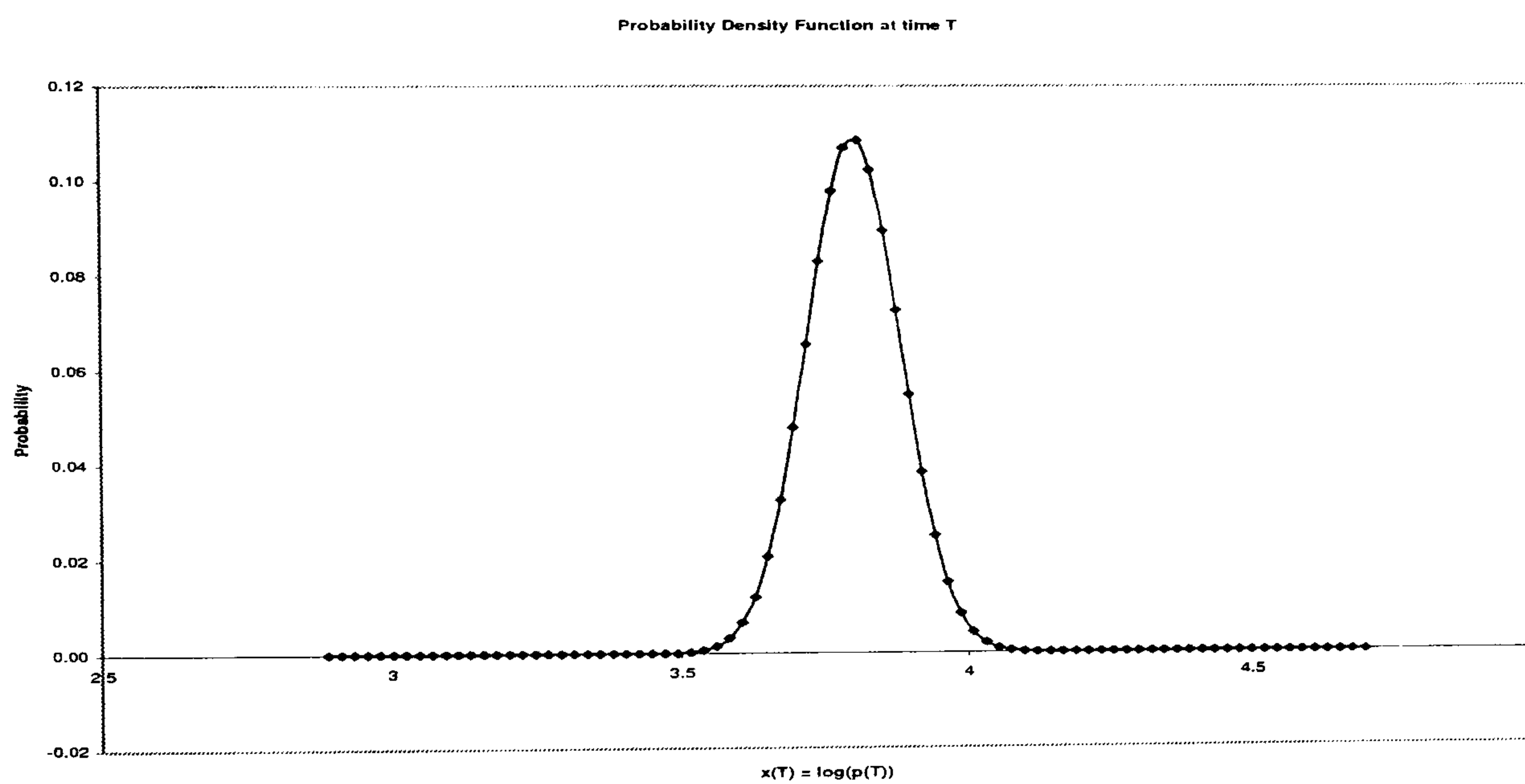


Figure 5.5: Probability density function for the spot prices sample paths at final time $T = 5$ for the unconstrained model.

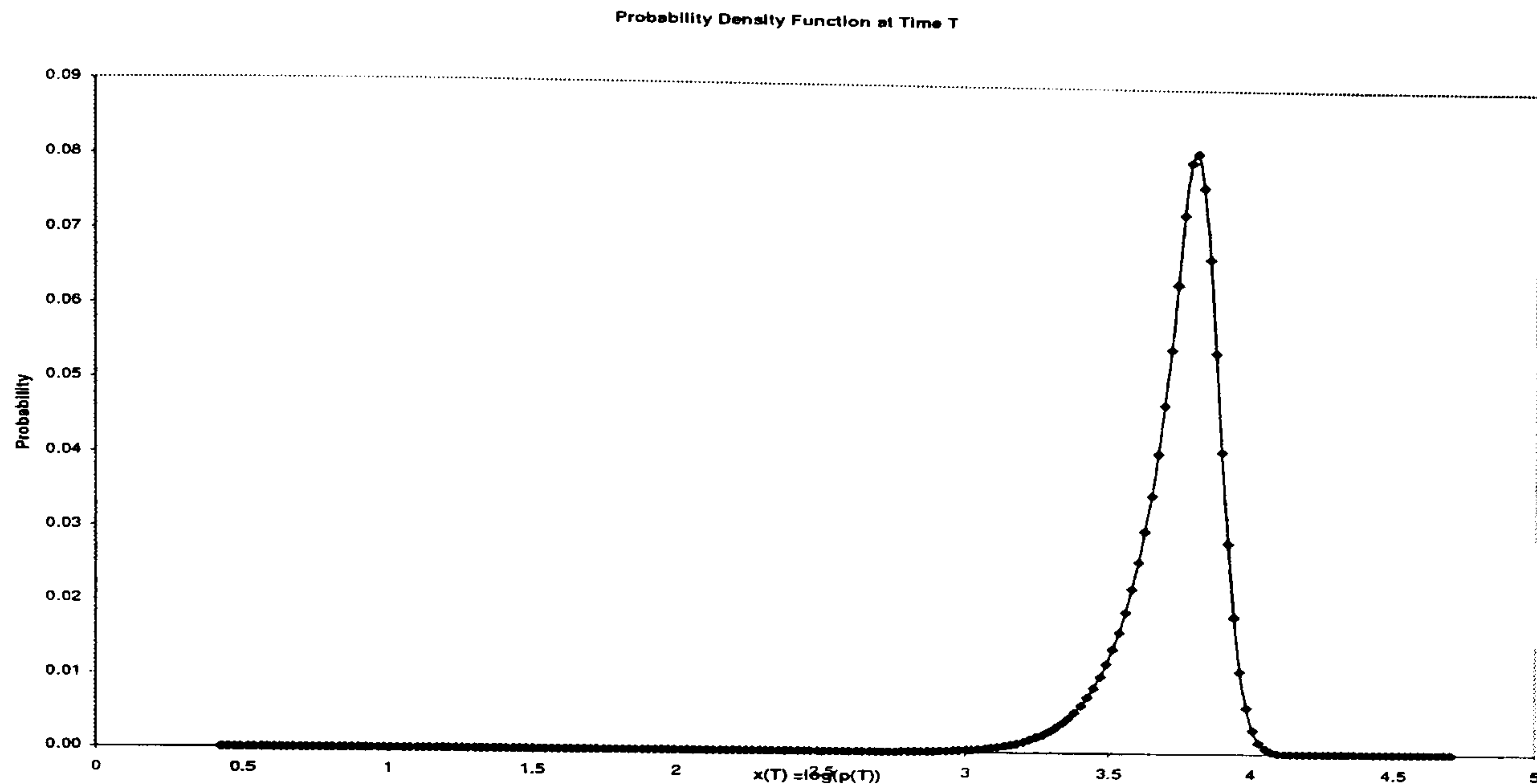


Figure 5.6: Probability density function for the spot prices sample paths at final time $T = 5$ for our model.

5.5 Conclusion

In this chapter, we have explored the properties of the simplest mean reverting model for commodity prices that satisfy the arbitrage-free contango constraint. Rather than modelling explicitly the convenience yield we impose a constraint in the evolution of the spot prices that rules out cash-and-carry arbitrage. Our model generates a rich set of forward curve dynamics. We observe backwardation when the spot prices are high and backwardation otherwise. In addition, the degree of backwardation (contango) is increasing (decreasing) with the initial spot prices. The slope of the forward curves in contango is considerable smaller (in absolute value) than the slope of the forward curves in backwardation since the contango is constrained. This generates asymmetry between the slopes of contango and backwardation forward curves. Our model also generates a leptokurtic distribution in the spot prices, which is typical in a storage economy. Particularly, when compared with the single-factor model described by Schwartz (1997) [82], our model intro-

duces two important features. First, it introduces excess kurtosis in the spot price distribution. Second, our model eliminates cash-and-carry arbitrage possibilities. We also suggest that other standard reduced form models such as Schwartz (1997) [82] two-factor model, Gibson and Schwartz (1990) [42], Hilliard and Reis (1998) [51] and Miltersen and Schwartz (1998) [68] may generate arbitrage since the convenience yield may become negative. In summary, while preserving arbitrage conditions, our model generates a rich set of price dynamics.

Nevertheless we recognize two main misspecifications in our model. First, the futures returns observed in contango are smaller than the cost of carrying inventory. This implies that if the initial spot price is very low when compared with the long-run mean price, the forward curve does not reach a steady-state equilibrium within a realistic length of time. This result diverges from the analysis of the forward curve provided in Chapters 3 and 4 and by Routledge Seppi and Spatt (2000) [78]. The other drawback is that the spot prices distribution is left skewed, which is not a desirable property in commodity price distributions. As explained before, commodity prices are characterized by long periods of stagnant prices interrupted by sudden upward spikes. This implies that the commodity spot price distributions have excess kurtosis and are right skewed. In particular, this right skewness is a consequence of the inventory non-negativity.

We suggest that further work should be pursued in order to improve the properties generated by this model. It would be desirable to build an arbitrage-free model that includes simultaneously the properties of commodity prices predicted by the theory of storage and, by the structural model developed in Chapters 3 and 4. It would be extremely valuable to develop a tractable reduced form model that generates an appropriate shape of forward curves and that preserves arbitrage-free conditions in commodity markets. In particular, it is important to take into

account a realistic definition of market volatilities rather than the constant volatility models in the literature. The commodity spot price volatilities are strongly heteroskedastic (see Duffie and Gray, 1995 [30]) and closely reflect the supply, demand and inventory conditions in the market. In particular, price volatilities are increasing with the degree of backwardation (see Ng and Pirrong, 1994 [72] and Litzenberger and Rabinowitz, 1995 [62]). The forward prices volatility are also complex and time-varying. Typically, the forward volatility decreases with time horizon - the Samuelson (1965) [79] effect. A correct definition of volatility of forward prices at different horizons is important for both derivative security pricing and dynamic hedging. Taking these properties into account we develop a new arbitrage-free reduced form model that includes time-varying spot price volatility in the next chapter.

Chapter 6

A Two-Factor Model for Commodity Prices and Futures Valuation

6.1 Introduction

In this chapter, we introduce a new reduced form model for commodity spot prices and futures valuation which builds on and extends the reduced form models in the literature.

The earliest reduced form model of a commodity price appears to be due to Brennan and Schwartz (1985) [10]. In this model the spot commodity price follows a geometric Brownian motion and the convenience yield is treated as a dividend yield. This specification is inappropriate since it does not take into account the mean reversion property of spot commodity prices and neglects the inventory-dependence property of the convenience yield. Gibson and Schwartz (1990) [42] and Schwartz (1997) [82] introduce a two-factor, constant volatility model where

the spot price and the convenience yield follow a joint stochastic process with constant correlation. Specifically, the spot price follows a geometric Brownian motion and the convenience yield follows a mean reverting stochastic process of the Ornstein-Uhlenbeck (O-U) type. The convenience yield is brought into the spot price process as a dividend yield. Schwartz (1997) [82] three-factor model, Miltersen and Schwartz (1998) [68] and Hilliard and Reis (1998) [51] add a third stochastic factor to the model to account for stochastic interest rates. Nevertheless, the inclusion of stochastic interest rates in the commodity price models does not have a significant impact in the pricing of commodity options and futures in practice. Accordingly, interest rate can be assumed deterministic.

The reduced form class of models dominates the current literature and practice on energy derivatives. These models are particularly attractive from practitioner's perspective since they provide closed form solutions to evaluate futures and some other derivatives contracts. This in turn allows for a relatively easy calibration and computational implementation of these models.

Although these multi-factor models generate a rich set of dynamics for the commodity term structure and represent prevailing tools for derivatives pricing, they also present a number of problems. First, these models do not guarantee that the convenience yield is always well defined and thus may become negative, possibly allowing for arbitrage opportunities. More specifically, arbitrage-free arguments require that the discounted futures prices net of carrying costs cannot be greater than the contemporaneous spot prices. By not ruling out negative values for the convenience yield, this arbitrage argument may be violated. Secondly, these models present other mis-specification problems due to the fact that both the spot price and the convenience yield have constant volatility and correlation. Accordingly, they do not allow the variance of the spot and futures, and the cor-

relation between them, to depend on the level of the price or convenience yield, as suggested by the theory of storage. The commodity spot price volatilities are strongly heteroskedastic (see Duffie and Gray, 1995 [30]) and closely reflect the supply, demand and inventory conditions in the market. In particular, price volatilities are increasing with the degree of backwardation (see Ng and Pirrong, 1994 [72] and Litzenberger and Rabinowitz, 1995 [62]). These mis-specifications may generate severe option mispricings, as pointed out by Pirrong (1998) [75], Clewlow and Strickland (2000) [16] and Routledge, Seppi and Spatt (2000) [78].

The reduced form model presented in this chapter extends the reduced form models of Gibson and Schwartz (1990) [42] and Schwartz (1997) [82]. More specifically, we develop a two-factor model where spot prices and instantaneous convenience yield follow a joint stochastic process with constant correlation. Our model introduces two significant additions to Schwartz (1997) [82] two-factor model: it rules out arbitrage possibilities and it considers time varying spot and convenience yield volatilities. Namely, the spot price follows a Geometric Brownian Motion (GBM) where the convenience yield is treated as an exogenous dividend yield and the volatility is proportional to the square root of the instantaneous convenience yield level. The instantaneous convenience yield follows a Cox-Ingersoll-Ross (CIR) which precludes negative values and makes the volatility proportional to the square root of the instantaneous convenience yield level. This ensures that our model does not allow cash-and-carry arbitrage possibilities.

We obtain a closed-form solution for the futures prices of the exponential affine form¹. We solve the partial differential equation (PDE) for futures prices by supposing that the solution has a general exponential affine form. By replacing this general affine form into the initial PDE we obtain a system of two ordinary

¹Since we assume that the interest rate is constant, the futures and forward prices are the same.

differential equations (ODEs) with boundary conditions consistent with the futures price expiry condition. We find that each of these ODEs has a unique closed form solution. These in turn provide the solution to the PDE satisfied by the futures prices². This affine relationship is tractable and offers empirical advantages. In particular, the linear relationship between the logarithm of the futures price and the underlying state variables allows the use of the Kalman filter in the estimation of the parameters of the model. Spot prices data are not easily obtained in most of the commodity markets and therefore futures prices with closest maturity are used as a proxy for the commodity price level. Additionally, the instantaneous convenience yield is not observable and must be derived from the relationship between the spot and futures prices with closest maturity. On the other hand, futures prices are widely observed and traded in diverse markets. The non-observability of the state variables remains one of the main difficulties in modelling commodity spot prices and contracts. Due to the non-observability of the state variables, the linearity of the logarithm of the futures prices in the state variables and the Markovian property of these, the Kalman filter seems to be the most appropriate technique to estimate the model's parameters. This method is also applied by Schwartz (1997) [82]. The basic principle of Kalman filter is the use of temporal series of observable variables to reconstitute the value of the non-observable variables. Accordingly, by observing futures prices, we can estimate the parameter values for the spot price and convenience yield.

We apply the Kalman filter method to estimate the parameters of our model using light crude oil futures data for the period from 17th of March 1999 to 24th of December 2003. Additionally, we also apply the Kalman filter to Schwartz (1997) [82] two factor model using the same data set for estimation purposes and to

²A similar solution method has been applied in the interest rate models literature such as Hull and White (1990) [54] Brown and Schaefer (1994) [11] and Duffie and Kan (1994, 1996) [31, 32].

compare the results. We also compare the futures volatility structure implied by the data and by both our model and Schwartz model.

The remaining chapter is organized as follows. Section 6.2 develops the two-factor model and derives the corresponding partial differential equation for futures valuation. Section 6.3 describes the empirical work, including the state-space formulation of the model, the data used and the empirical results. Section 6.4 concludes.

6.2 Valuation Model

In this section we present the commodity price model and derive the corresponding formulas for pricing futures contracts. This model has two stochastic factors. The first factor is the spot price, which follows as a GBM with a time-varying volatility, which is proportional to the square root of the instantaneous convenience yield level. The second factor is the convenience yield, which follows a CIR stochastic process as described by Cox, Ingersoll and Ross (1985) [23]. This process precludes negative convenience yields and implies that the absolute variance of the convenience yield increases when the convenience yield itself increases. We assume that both stochastic processes have constant correlation. The direct proportionality of the spot price and the convenience yield volatilities to the square root of the instantaneous convenience yield reflect the effect of supply, demand and inventory market conditions on the spot price and the convenience yield volatilities. As Duffie and Gray (1995) [30] point out, the commodity spot price volatilities are strongly heteroskedastic and closely reflect the supply, demand and inventory conditions in the market. In particular, price volatilities are increasing with the degree of backwardation (see Ng and Pirrong, 1994 [72] and Litzenberger and Rabinowitz, 1995 [62]). That is, the stronger the backwardation is the higher

the convenience yield is. High convenience yield levels signal low inventory and the possibility of a stockout. Therefore, the spot price volatility is positively related with the value of the convenience yield. Similarly, when the market is in contango, the spot prices volatility should be low. The market conditions and the commodity spot prices also affect similarly the volatility of the convenience yield itself .

The model we present is very tractable since it allows a closed form solution to futures prices. Specifically, we obtain a linear relation between the logarithm of futures prices and the underlying factors. This property is crucial to the empirical work that follows.

We assume that the spot price and the instantaneous convenience yield follow the joint stochastic process:

$$dp = \mu(\cdot)pdt + \sigma_1\sqrt{\delta}pdB_1 \quad (6.1)$$

$$d\delta = (\alpha(m - \delta))dt + \sigma_2\sqrt{\delta}dB_2 \quad (6.2)$$

where:

- $\mu(\cdot)$ is the total expected return on the spot commodity price;
- σ_1 represents the constant of proportionality between the total spot price volatility and the square root of the instantaneous convenience yield;
- σ_2 represents the constant of proportionality between the total instantaneous convenience yield volatility and the square root of the instantaneous convenience yield;
- α is the instantaneous convenience yield's speed of mean reversion;

- m is the convenience yield long-run mean, that is, the level to which δ reverts as t goes to infinity;
- B_1 and B_2 are standard Wiener processes and are correlated with $dB_1dB_2 = \rho dt$, ρ being constant.

The probability density of the convenience yield at time t conditional on its value at current time t is a non-central chi-square (see Cox, Ingersoll and Ross, 1985 [23]). The conditional moments of δ at time t under the objective measure are given by:

$$E[\delta_t|\delta_{t-dt}] = m(1 - e^{-\alpha dt}) + \delta_{t-dt}e^{-\alpha dt}, \quad (6.3)$$

$$Var[\delta_t|\delta_{t-dt}] = m \left(\frac{\sigma_2^2}{2\alpha} \right) (1 - e^{-\alpha dt})^2 + \delta_{t-dt} \left(\frac{\sigma_2^2}{\alpha} \right) (e^{-\alpha dt} - e^{-2\alpha dt}). \quad (6.4)$$

By defining $x = \ln p$ and applying Ito's Lemma, the process for the log price is given by:

$$dx = \left(\mu(\cdot) - \left(1 + \frac{1}{2}\sigma_1^2 \right) \delta \right) dt + \sigma_1 \sqrt{\delta} dB_1. \quad (6.5)$$

Under the risk neutral measure, the stochastic processes that drive the the state variables becomes:

$$dp = (r + c - \delta)pdt + \sigma_1 \sqrt{\delta} p dB_1^*, \quad (6.6)$$

$$d\delta = (\alpha(m - \delta) - \lambda_\delta)dt + \sigma_2 \sqrt{\delta} dB_2^*, \quad (6.7)$$

where:

- r is the risk-free (constant) interest rate;
- c is the (constant) marginal cost of storage, which is a proportion of the spot price;
- λ is the (constant) market price of risk for the convenience yield;
- σ_1 , σ_2 , α and m are as before;
- B_1^* and B_2^* are standard Wiener processes under the risk-neutral measure and are correlated with $dB_1^*dB_2^* = \rho dt$, ρ as before.

The expected growth of the commodity price in a risk-neutral world is $\mu - \lambda_p \sigma_1$, where λ_p is the market price of risk of commodity price. Since the commodity behaves like a traded security that provides a dividend rate equal to δ , the expected growth rate of the commodity price under the risk-neutral measure is also given by $r + c - \delta$. Therefore $r + c - \delta = \mu(\cdot) - \lambda_p \sigma_1 \sqrt{\delta}$. Accordingly, the drift of the spot price process, $\mu(\cdot)$ in the real measure is replaced by $(r + c - \delta)$ under the risk neutral measure. Equation (6.6) is an extension of a standard process for the commodity process allowing for a stochastic convenience yield and a time varying volatility. This volatility is proportional to the square root of the time-varying stochastic convenience yield. On the other hand, since the convenience yield is non-traded, the convenience yield risk cannot be hedged and will have a market price of risk, λ_δ associated with it. Therefore, the drift of the convenience yield under the risk neutral measure becomes $[\alpha(m - \delta) - \lambda_\delta]$. Although λ_δ is a function of $\alpha(m - \delta)$ and $\sigma_2 \sqrt{\delta}$, we assume that the convenience yield market price of risk is constant. This assumption is standard and is also followed by Schwartz (1997)

[82]. The process for the log price then becomes:

$$dx = \left(r + c - \left(1 + \frac{1}{2}\sigma_1^2 \right) \delta \right) dt + \sigma_1 \sqrt{\delta} dB_1^*. \quad (6.8)$$

By assuming that the instantaneous convenience yield follows a CIR process we ensure that our model is arbitrage-free because it precludes negative values. This assumption ensures that the discounted futures prices net of carrying costs cannot be greater than the discounted contemporaneous spot prices, which is derived by arbitrage-free arguments. Considering that $\tau = T - t$ represents time to maturity the arbitrage-free condition can be written as:

$$F(\tau) \leq p_t \exp\{(r + c)(\tau)\}, \quad (6.9)$$

where

- $F(\tau)$ is the forward price at time t , for delivery of a commodity at time $T > t$;
- p_t is the spot price of the commodity at time t ;
- c is the (constant) proportion of the spot price which defines the marginal cost of storage;
- r is risk-free (constant) interest rate.

If the instantaneous convenience yield is always non-negative the arbitrage-free

condition in equation (6.9) is satisfied³.

The futures prices must satisfy the partial differential equation (PDE), which is fully derived in Appendix F:

$$\begin{aligned} \frac{1}{2}\sigma_1^2\delta p^2 F_{pp} + \frac{1}{2}\sigma_2^2\delta F_{\delta\delta} + \rho\sigma_1\sigma_2\delta p F_{p\delta} + ((r+c)-\delta)p F_p + \\ (\alpha(m-\delta)-\lambda) F_\delta - F_\tau = 0 \end{aligned} \quad (6.10)$$

subject to the boundary condition $F(p, \delta, 0) = p$. This PDE suggests an exponential affine form solution:

$$F(p, \delta, \tau) = p e^{A(\tau) - B(\tau)\delta}, \quad (6.11)$$

with

$$A(0) = 0; \quad B(0) = 0. \quad (6.12)$$

Equivalently, the logarithm of the futures prices is given by:

$$\ln F(p, \delta, \tau) = \ln p + A(\tau) - B(\tau)\delta. \quad (6.13)$$

The futures prices as given by equation (6.11) satisfy the PDE (6.10) and the boundary condition when

$$\frac{1}{2}\sigma_2^2 B^2 + (\alpha - \rho\sigma_1\sigma_2)B - 1 + B_\tau = 0, \quad (6.14)$$

³Note that Schwartz (1997) [82] does not ensure non-negative convenience yield given that the stochastic convenience yield in his two-factor model follows an Ornstein-Uhlenbeck. This may generate arbitrage possibilities in his model.

and

$$(r + c) + (\lambda - \alpha m)B - A_\tau = 0, \quad (6.15)$$

with initial conditions

$$A(0) = 0; \quad B(0) = 0. \quad (6.16)$$

It follows that if (6.14) and (6.15) are solved subject to the boundary conditions in (6.16), equation (6.11) provides the price of a futures contract maturing at time T . Appendix G provides the derivation of the ODEs and respective solution, which is given by:

$$B(\tau) = \frac{2(1 - e^{-k_1\tau})}{k_1 + k_2 + (k_1 - k_2)e^{-k_1\tau}} \quad (6.17)$$

and

$$A(\tau) = (r + c)\tau + (\lambda - \alpha m) \int_t^T B(q)dq, \quad (6.18)$$

where:

$$\int_t^T B(q)dq = \frac{2}{k_1(k_1 + k_2)} \ln \left[\frac{(k_1 + k_2)e^{k_1\tau} + k_1 - k_2}{2k_1} \right] + \quad (6.19)$$

$$\frac{2}{k_1(k_1 - k_2)} \ln \left[\frac{k_1 + k_2 + (k_1 - k_2)e^{-k_1\tau}}{2k_1} \right], \quad (6.20)$$

with:

$$k_1 = \sqrt{k_2^2 + 2\sigma_2^2} \quad (6.21)$$

$$k_2 = (\alpha - \rho\sigma_1\sigma_2) \quad (6.22)$$

The solution to equation (6.10) with initial boundary condition $F(p, \delta, 0) = p$ is given by (6.11) with $A(\tau)$ and $B(\tau)$ given by (6.17) and (6.18).

6.3 Empirical Estimation of the Joint Stochastic Process

In this section we estimate and empirically test both our model and the Schwartz (1997) [82] two-factor model. Data for most of the spot commodity prices are extremely difficult to obtain price for most of the commodities. On the other hand, we are able to observe daily several futures prices at different maturities. This non-observability and the linear relationship between futures prices and the state variables in the model clearly suggest that the Kalman filter is the most appropriate technique to estimate the model's parameters.

The principle of Kalman filter is to use a time series of observable variables and to infer the value of the non-observable variables. This technique is suitable whenever there is a linear dependency of the observable variables upon the state variables and when the later are Markovian processes. Kalman filter is a technique which has become increasingly popular in Finance and has been applied to both Gaussian and CIR type interest rate models and in commodity futures valuation in (e.g. Schwartz, 1997 [82] and Schwartz and Smith, 2000 [84]). Affine models are particularly suited for estimating using Kalman filter because of their linear structure. In the context of interest rate models Gaussian examples can be found in Babbs and Nowman (1999) [2] and Lund (1997) [63], who estimate a two-factor generalized Vasicek model. In the CIR case, there are examples due to Ball and Torous (1996) [3], Duan and Simonato (1995) [29] and Lund (1997) [63]. Schwartz (1997) [82] two-factor model belongs to the Gaussian class while our model fits in the CIR class.

The state form is applied to a multivariate time series of observable variables, which in our case are a futures prices time series at several different ma-

turities. These observed variables are related to the state vector which consist of the state variables, which in our model are the spot price and the instantaneous convenience yield via the *measurement equation*. The measurement equation is then given by equation (6.13) by adding serially and cross sectionally uncorrelated disturbances with mean zero and variance to take into account for the irregularities of the observations. In the Kalman filter, the non-observable state variables are generated by first-order Markov processes which correspond to the discrete time form of equations (6.1) and (6.2). The latter are arranged in a vector, which forms the *transition equation*. See Harvey (1989)[48] for a detailed description of this method. We calibrate Schwartz (1997) [82] two-factor⁴ model using exactly the same methodology as described in his article. To calibrate our model we follow the same steps but we need to take into account an important difference between the two empirical models. The state-space form of Schwartz's model is Gaussian while the state-space form of our model is non-Gaussian, given that we do not have constant volatility.

For a Gaussian state-space model, the Kalman filter provides an optimal solution to prediction, updating and evaluating the likelihood function. The Kalman filter recursion is a set of equations which allows an estimator to be updated once a new observation becomes available. The Kalman filter first forms an optimal predictor of the unobserved state variable vector given its previously estimated value. This prediction is obtained using the distribution of unobserved state variables, conditional on the previous estimated values. These estimates for the unobserved state variables are then updated using the information provided by the observed

⁴According to Schwartz (1997) [82] two-factor model, the commodity spot price and the convenience yield follow a joint stochastic process with constant correlation given by:

$$\begin{aligned} dp &= \mu(\cdot)pdt + \sigma_1 p dB_1, \\ d\delta &= (\alpha(m - \delta))dt + \sigma_2 dB_2, \end{aligned}$$

where $dB_1 dB_2 = \rho dt$.

variables. Prediction errors, obtained as a by-product of the Kalman filter, can then be used to evaluate the likelihood function.

When the state-space model is non-Gaussian, the Kalman filter can still be applied and the resulting filter is quasi optimal. This filter is then used to obtain a quasi-likelihood function and the estimates obtained is linearly optimal. This approximation is needed because of the non-Gaussian nature of the problem, which can be compared to linearizing a non-linear function in the typical Kalman filtering applications. Duan and Simonato (1995) [29] and Geyer and Pichler (1998) [41] apply Kalman Filter to estimate and test exponential-affine term structure models for both the Gaussian and non-Gaussian cases. For a detailed discussion see Duan and Simonato (1995) [29] and Harvey (1989) [48].

It is also important to mention that the CIR process also differs from standard Kalman filter application because of the non-negative constraint on the convenience yield. Following Duan and Simonato (1995) [29] and Geyer and Pichler (1998) [41] we modify the standard Kalman filter by simply replacing any negative element of the convenience yield estimate with zero.

6.3.1 State Space Formulation

From the valuation formula given by equations (6.13), (6.17) and (6.18), the measurement equation can be written as:

$$Y_t = d_t + Z_t[x_t, \delta_t]' + \varepsilon_t, \quad t = 1, \dots, N \quad (6.23)$$

where:

- $Y_t = [\ln F(\tau_i)]$, for $i = 1, \dots, n$ is a $n \times 1$ vector of observations where $F(\tau_i)$ is the observed futures price at time t for maturity τ_i . At each time t we observe n futures prices which correspond to n different maturities;

- $d_t = [A(\tau_i)]$ for $i = 1, \dots, n$ is a $n \times 1$ where $A(\cdot)$ is given by equation (6.18).
- $Z_t = [1, -B(\tau_i)]$, for $i = 1, \dots, n$ is a $n \times 2$ matrix where $B(\cdot)$ is calculated according to equation (6.17);
- ε_t is a $n \times 1$ is $n \times 1$ vector of serially uncorrelated disturbances with $E[\varepsilon_t] = 0$, $Var[\varepsilon_t] = H_t$. This vector is introduced to account for possible errors in the data. The covariance matrix H_t is taken to be diagonal for computational simplicity;

The transition equation is given by:

$$[x_t, \delta_t]' = c_t + Q_t[x_t, \delta_t]' + R_t\eta_t, \quad t = 1, \dots, NT \quad (6.24)$$

where:

- $$c_t = \begin{bmatrix} \mu\Delta t \\ m(1 - e^{-\alpha\Delta t}) \end{bmatrix} \quad (6.25)$$

- $$Q_t = \begin{bmatrix} 1 & -(1 + \frac{1}{2}\sigma_1^2)\Delta t \\ 0 & e^{-\alpha\Delta t} \end{bmatrix} \quad (6.26)$$

- R_t is a 2×2 identity matrix;
- η_t is a 2×1 vector of serially uncorrelated disturbances with $E[\eta_t] = 0$ and $Var[\eta_t] = V_t$.

The covariance matrix of η_t is given by:

$$V_t = \begin{bmatrix} \sigma_1^2 \Delta t \delta_{t-dt} & \rho\sigma_1 \sqrt{\Delta t} \sqrt{\delta_{t-dt}} \sqrt{Var[\delta_t|\delta_{t-1}]} \\ \rho\sigma_1 \sqrt{\Delta t} \sqrt{\delta_{t-dt}} \sqrt{Var[\delta_t|\delta_{t-1}]} & Var[\delta_t|\delta_{t-1}] \end{bmatrix} \quad (6.27)$$

where:

$$Var[\delta_t|\delta_{t-1}] = m \left(\frac{\sigma_2^2}{2\alpha} \right) (1 - e^{-\alpha\Delta t})^2 + \delta_{t-1} \left(\frac{\sigma_2^2}{\alpha} \right) (e^{-\alpha\Delta t} - e^{-2\alpha\Delta t}) \quad (6.28)$$

The observation and state equation matrices Z_t , d_t , H_t , Q_t , c_t and V_t depend on the unknown parameters of the model. One of the main purposes of the Kalman filter implementation is to find estimates for these parameters. This can be done by maximizing the quasi likelihood function with respect to the unknown parameters through an optimization procedure. In appendix H we present further details on the Kalman filter implementation procedure.

For notational simplicity, consider θ the vector of unknown parameters and $Y_t = \{y_t, y_{t-\Delta t}, \dots, y_t, y_{t_0}\}$ the information vector at time t , which are not independent. We assume that the distribution of Y_t conditional on $Y_{t-\Delta t}$ under the objective measure is normal with mean $\hat{Y}_{t|t-\Delta t} = E[Y_t|Y_{t-\Delta t}]$ and covariance matrix F_t . The vector of prediction errors is given by $v_t = Y_t - \hat{Y}_{t|t-\Delta t}$. The logarithm of the *quasi-likelihood function* is given by:

$$\log L(Y; \theta) = -\frac{1}{2} \frac{n(t_{final} - t_0)}{\Delta t} \log 2\pi - \frac{1}{2} \sum_t \log |F_t| - \frac{1}{2} \sum_t v_t F_t^{-1} v_t \quad (6.29)$$

Since both F_t and v_t depend upon θ , θ_t is chosen to maximize the quasi-likelihood function⁵. This estimation procedure is recursive and it is calculated at each time t as part of the Kalman filter.

To calibrate Schwartz (1997) [82] two-factor model we use the state-space formulation as described in his paper.

⁵To maximize the quasi-likelihood function we used the Matlab routine "maxlik.m". This function is part of the Econometrics Toolbox by James P. LeSage and can be downloaded for free at: www.spatial-econometrics.com.

6.3.2 Empirical Results

The data set used in this study consists of weekly observations of New York Mercantile Exchange (NYMEX) light crude oil futures which covers the period from 17th of March 1999 to 24th of December 2003 (243 observations)⁶. At each observation we consider 7 contracts ($n = 7$) corresponding to 7 different maturities. Naturally, the time to maturity changes as we evolve in time and to force the time to maturity to stay within a narrow range we roll over the contracts during the period of observations. Table 6.1 describes the data used. We denote by F0 the contract closest maturity, F1 the second contract closest to maturity and so on. We assume that the interest rate, r , is equal to 0.04 and the marginal storage cost of storage is equal to 0.20.

Futures Contract	Mean Maturity (Standard Deviation)	Mean Price (Standard Deviation)	Volatility of Returns
F0	0.044 (0.024)	27.058 (4.711)	0.377
F1	0.127 (0.025)	26.724 (4.435)	0.346
F2	0.348 (0.024)	25.712 (3.884)	0.273
F3	0.598 (0.024)	24.735 (3.514)	0.231
F4	0.931 (0.024)	23.759 (3.173)	0.191
F5	1.181 (0.024)	23.189 (2.974)	0.176
F6	1.931 (0.024)	21.963 (2.599)	0.158

Table 6.1: Light Crude Oil Futures weekly data from 17th of March 1999 to 24th of December 2003.

Table 6.2 reports the estimation results for both our model and Schwartz

⁶The data was retrieved from the Internet on the 31st of October 2003 and on the 17th of February 2004 from *Futures Guide*TM, <http://www.futuresguide.com/index.php>. The original data set consists of daily observations. Weekly data was obtained by using every Wednesday (to avoid weekend effects) observation.

(1997) [82] two-factor model. The values obtained for the parameters are comparable for both cases. The most noticeable difference lies in the value of the long-run mean for the convenience yield, m . However, this difference is approximately 0.2 which is consistent with the storage cost value of 0.2 that we assume in our model.

The speed of mean reversion in the convenience yield equation, α , and the coefficient of correlation between the spot price and convenience yield, ρ , are high and significant for both cases. The total expected return on the spot commodity, μ , and the market price of convenience yield, λ , are also positive and high. In particular, it is worth mentioning the high value of average convenience yield. This indicates that during this period the market is predominantly in backwardation. Additionally, both spot price and convenience yield volatilities are also high. This behavior in the crude oil market can be explained by the world events which took place after September 2001 and, in particular, the recent Gulf war. These events lead to increasingly uncertainty in the world markets in general and in particular in the oil supply. This uncertainty naturally rises the value of having crude oil in storage, which implies a high convenience and a market in strong backwardation.

As mentioned before, we assume a diagonal covariance structure for the measurement errors. These are denoted by v_0 , v_1 , v_2 , v_4 , v_4 and v_5 and v_6 and correspond to each of the futures contracts used, namely $F0$, $F1$, $F2$, $F3$, $F4$, $F5$ and $F6$ respectively. These values are also displayed in table 6.2. The magnitude of this errors is the same for both our model and Schwartz (1997) [82] two-factor model.

Parameters	Our model	Schwartz's (1997)[82] two-factor model
m	0.526 (0.011)	0.356 (0.011)
λ	1.617 (0.008)	1.869 (0.002)
α	6.301 (0.019)	6.273 (0.002)
σ_1	0.434 (0.005)	0.441 (0.001)
σ_2	0.725 (0.004)	0.720 (0.000)
ρ	0.899 (0.003)	0.800 (0.001)
μ	0.514 (0.006)	0.500 (0.000)
v_0	0.051 (0.001)	0.050 (0.002)
v_1	0.048 (0.001)	0.052 (0.003)
v_2	0.041 (0.001)	0.036 (0.001)
v_3	0.034 (0.001)	0.028 (0.001)
v_4	0.033 (0.001)	0.026 (0.001)
v_5	0.041 (0.002)	0.031 (0.002)
v_6	0.056 (0.003)	0.040 (0.002)
Log-likelihood function	3718	3597

Table 6.2: Estimation results and standard errors (in parentheses) for both our model and Schwartz (1997) [82] two-factor model using all the futures contracts F0, F1, F2, F3, F4, F5 and F6 from 17th of March 1999 to 24th of December 2003.

Table 6.3 displays the mean pricing errors (MPE) and the root mean squared errors (RMSE) for all the observations. Both error measures are small and of the same order of magnitude for both our model and Schwartz model. The values of the log-likelihood function and the pricing errors indicate that our model fits better

the data but by only a small amount. We also note that the performance of both models decreases as maturity of the futures contract increases. This highlights the fact that both models become less efficient as we increase the maturity of the futures contracts.

Figures 6.1 and 6.2 illustrate examples of the evolution of the forward curve of the market prices and both models.

Table 6.4 and Figure 6.3 display the volatilities implied by the market, our model and Schwartz (1997) [82] two-factor model. For short maturities both models underestimate the market volatilities and for longer maturities, these models increasingly overestimate them. This indicates that neither model is able to fit accurately the market volatility term structure. This certainly has implications in the valuation of financial or real asset contingent on a commodity price.

	Our Model		Schwartz (1997) Two-factor Model	
Futures Contract	RMSE	MPE	RMSE	MPE
F0	0.023	0.017	0.021	-0.001
F1	0.022	-0.017	0.034	-0.020
F2	0.044	-0.036	0.034	-0.023
F3	0.027	-0.019	0.014	-0.007
F4	0.020	0.000	0.038	0.013
F5	0.033	0.015	0.034	0.024
F6	0.059	0.029	0.060	0.020
Total	0.033	-0.002	0.034	0.002

Table 6.3: Summary statistics both our model's and Schwartz (1997) [82] two-factor model's pricing errors in valuing futures contracts during the whole period 17th of March 1999 to 24th of December 2003.

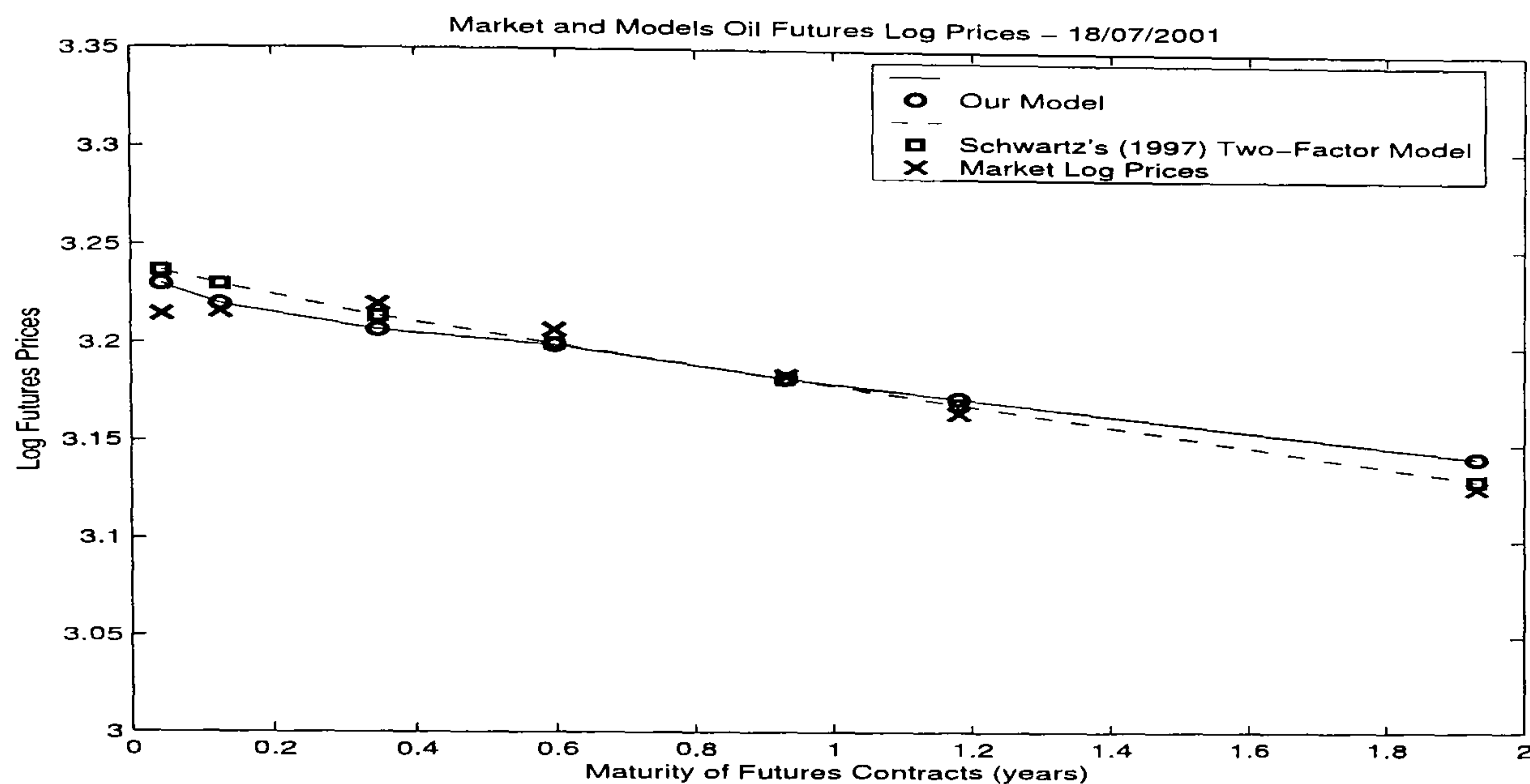


Figure 6.1: This figure illustrates the evolution of the forward curve for the market of futures prices and both our model and the Schwartz (1997) [82] two-factor model on the 5th of November 2002.

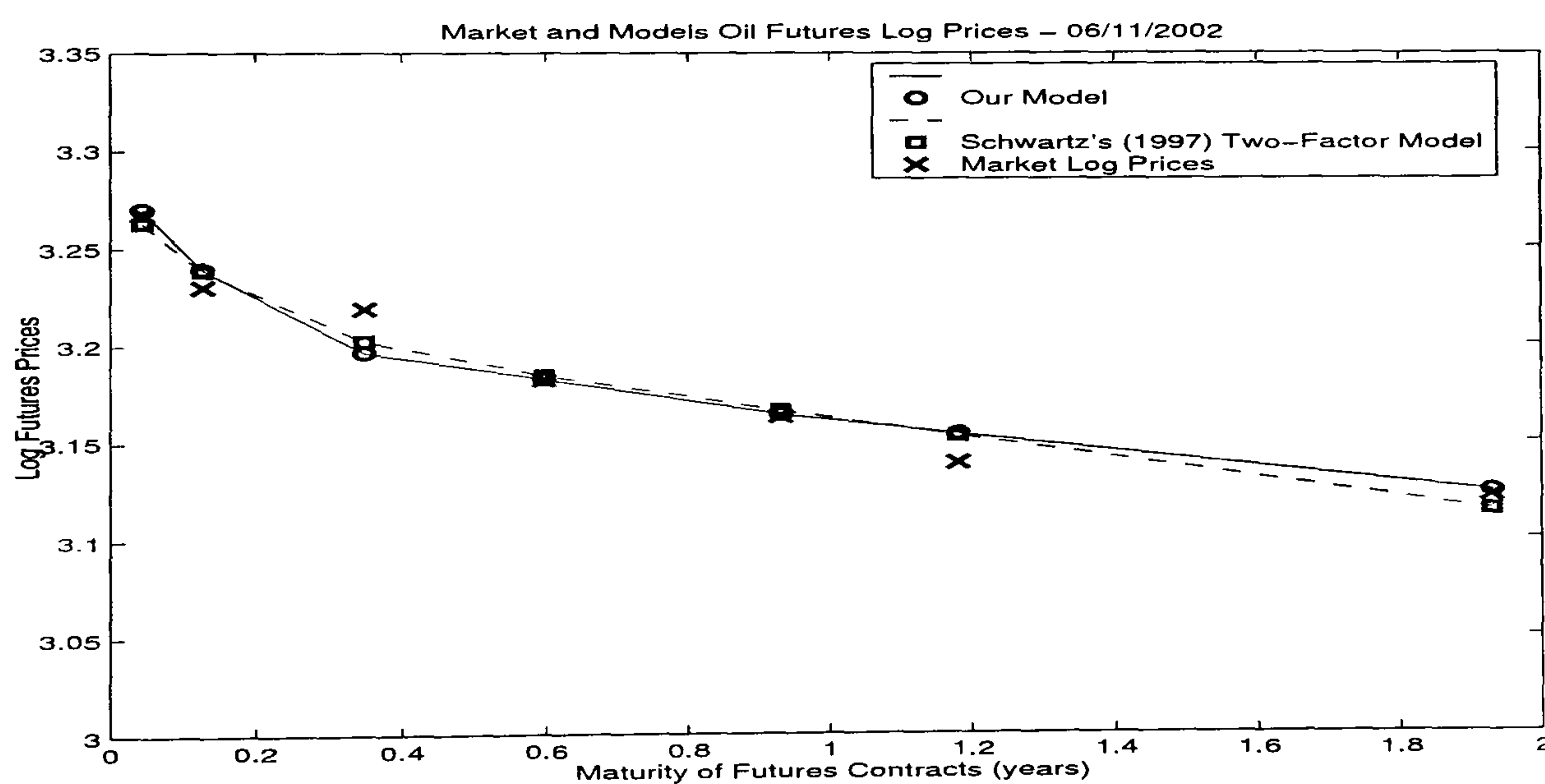


Figure 6.2: This figure illustrates the evolution of the forward curve for the market of futures prices and both our model and the Schwartz (1997) [82] two-factor model on the 18th of July 2001.

Futures Contract	Market Volatility	Our Model Volatility	Schwartz's Model Volatility
F0	0.377	0.386	0.330
F1	0.346	0.306	0.274
F2	0.273	0.234	0.219
F3	0.231	0.219	0.209
F4	0.191	0.216	0.207
F5	0.176	0.216	0.207
F6	0.158	0.216	0.207

Table 6.4: Market, our model and Schwartz (1997) [82] Two-Factor Model implied volatilities of annualized log-returns of futures prices.

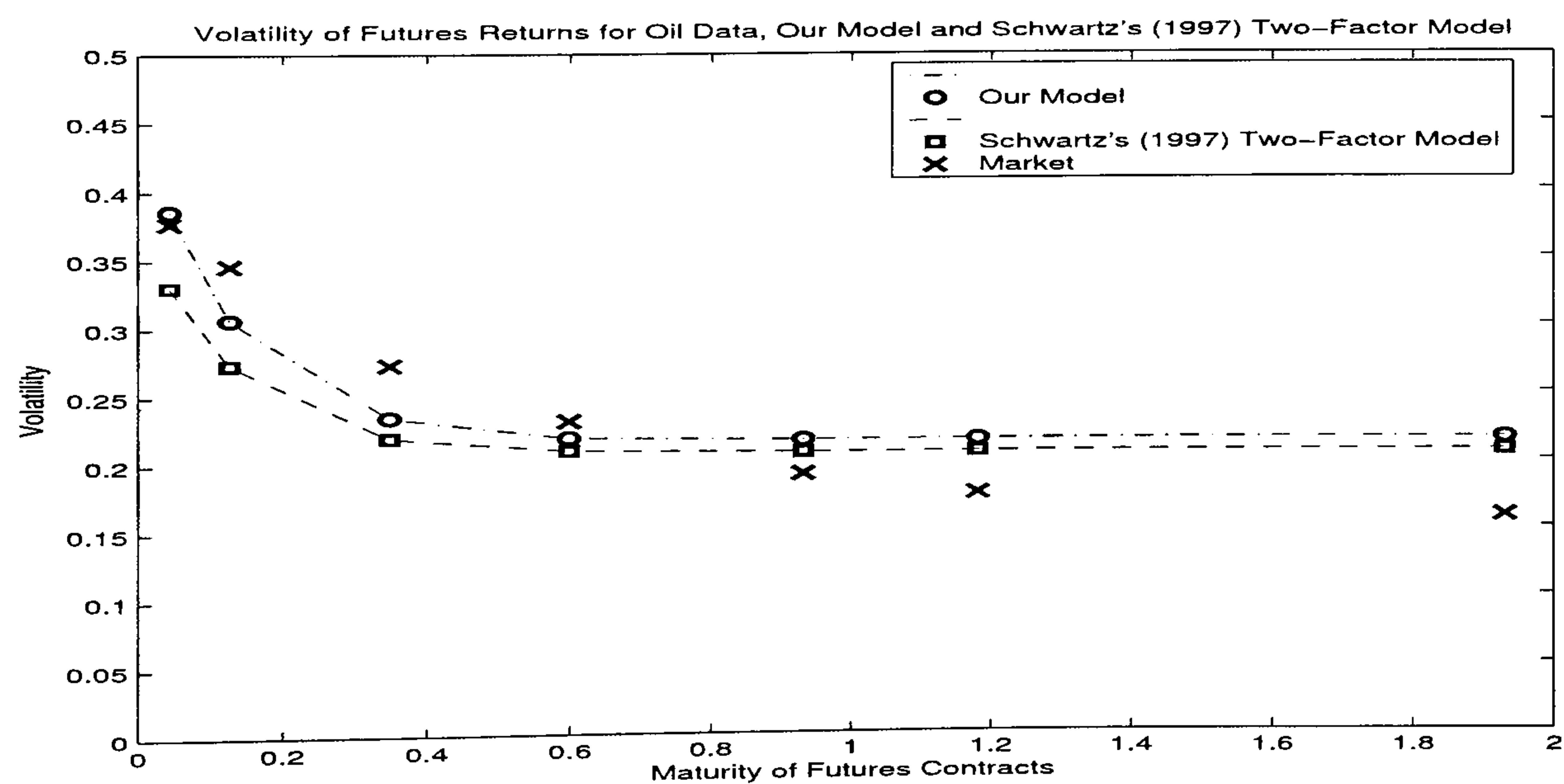


Figure 6.3: This figure illustrates the annualized volatility of futures returns implied by our model, Schwartz (1997) [82] Two-Factor Model and the market data.

6.4 Conclusion

In this chapter we presented a two-factor model for commodity prices and the corresponding futures valuation. This model extends Schwartz (1997) [82] two factor model by adding two important features. First, the O-U process for the convenience yield is replaced by a CIR process. This allows us to maintain the mean-reverting property of the convenience yield and to additionally ensure that our model is arbitrage-free. Second, we consider that both the spot price and the convenience yield volatilities are proportional to the square root of the instantaneous convenience yield level. This specification establishes a dependency between the commodity price volatility and the inventory levels. As explained before, this property is predicted by the theory of storage and by the structural models in the literature. This characteristic is also illustrated by the structural model developed in Chapters 3 and 4.

We derived the analytical solution for the futures partial differential equation. By guessing a solution of the exponential affine form, we transformed the initial PDE into a system two ODEs with analytical solution. This solution also satisfies the original PDE with the appropriate boundary conditions. This method has been also applied in the interest models literature such as Hull and White (1990) [54], Brown and Shaefer (1994) [11] and Duffie and Kan (1994, 1996) [31, 32].

We implemented empirically both our model and Schwartz (1997) two factor model for purposes of comparison using crude oil futures prices applying the Kalman filter. We found that the parameters estimates are analogous for both models and indicate a strong mean reversion in the convenience yield and strong backwardation. Both the 11th of September event and the recent war crisis might explain such behavior in the oil prices.

Although our model adds valuable characteristics to the existing reduced form models in the literature, it only outperforms Schwartz model by a small amount, in terms of both the pricing errors and the value of the log-likelihood function. Both models achieve very good results when valuating short-term maturity data but their efficiency decreases when the futures maturity increases. Neither can they reproduce the empirical volatility structure accurately. Both models underestimate short-term volatilities and over estimate longer term ones. This certainly has implications on contingent claims pricing for which the volatility is a fundamental element. Therefore, this study suggests that both our model and Schwartz (1997) [82] models should be extended in order to reproduce accurately the volatility structure of the commodity forward curves. This is particularly relevant to evaluate long-term investments on commodities.

With hindsight, we believe that the results were affected by the peculiarity of the data set used. In particular, we do not observed contango in the whole data set available to us. One of the advantages of our model in relation to Schwartz (1997) [82] model is the exclusion of arbitrage possibilities when the forward curve is in contango. Therefore, the empirical analysis of this particular data set did not allow us to illustrate this advantage. Accordingly, one of the directions of further work is to test both models for a different commodity or for different time period. Another direction for future work is the extension of this analysis to price commodity options in order to study the implications of this model in option pricing.

Chapter 7

Conclusion

7.1 Summary and Contributions

This thesis developed innovative continuous time models for the price of a storable commodity. The significance of this study is highlighted by the recent deregulation of energy markets worldwide and the resulting rapid expansion of energy derivatives trading. The review of the literature provided in Chapter 2 demonstrated that the current literature does not provide efficient models for commodity prices and futures valuation. We also pointed out that these misspecifications have important implications for derivatives pricing, which generates options mispricings. We claimed that this inadequacy is partly due to the fact that the two main streams in the literature - structural models and reduced form models - are largely disjoint. We showed that there are three crucial aspects within each of these classes of models that highlight this separation. The first is that the structural models do not take into account the mean-reverting properties of commodity prices. In contrast, this property is a central feature of the reduced form models in the literature and it is a fundamental property of commodity prices. The second is that the specification of the current reduced form models does not ex-

clude cash-and-carry arbitrage possibilities. Remarkably, the reduced form models literature has been unconcerned about this issue. On the contrary, this arbitrage free condition is one of the central features in the specification of the structural models. The third is that reduced form models do not take into account the dependence of spot price volatilities on the supply, demand and inventory conditions in the market. This property is largely advocated by the theory of storage and by structural models for commodity prices.

Accordingly, one of the main objectives of this study is to improve and to expand leading models within each of these streams of the literature. In doing so, we focused on the gaps between these two classes of models and established a link between them. In particular, we developed a structural model that takes into account the mean reversion of commodity prices. Taking the properties implied by this structural model and by the theory of storage into account, we developed two reduced form models. One of these models focuses both on the mean reverting properties of commodity prices and carefully excludes cash-and-carry arbitrage possibilities. The other model is more sophisticated since it also takes into account the dependence of the spot price volatility on the inventory levels as it is observed in structural models and proposed by the theory of storage.

In Chapter 3 we developed a new stochastic structural model in continuous time for the price of a storable commodity. In doing so, we developed the model under a general framework and provided a separate analysis of the competitive and monopolistic storage markets. We applied stochastic continuous time dynamic programming techniques in order to obtain a numerical solution for the optimal storage policy, the price dynamics and the price variability. We provided and compared numerical examples for each of the competitive and the monopolistic storage economies. The model presented in Chapter 3 makes several contributions

to the current literature. First, it introduces a continuous time structural model that draws on specific microeconomics assumptions of the market environment and establishes a link with the existing reduced form models. That is, it builds on the structural models, but it uses a continuous time framework and includes the mean reverting characteristics of commodity prices. The latter is particularly relevant since none of the existing structural models has included the mean reversion property of commodity prices. Second, this model is developed under a very flexible framework which allows for different extensions of the model to be adapted to different commodities. For example, this model can be extended to accommodate other type of supply/demand functions. In particular, we explained how seasonality could be included in the model without adding extra complexity to the solution method. Third, we formulated this model for both the competitive storage and the monopolistic storage environments and provided a valuable comparison between competitive and monopolistic storage policies and how these differences are reflected in the price dynamics. This comparison is not illustrated in the current storage models, which only consider a competitive storage economy. Since the energy markets have evolved from a monopolistic to a competitive environment in recent years, we stress the importance to analyze both storage economies. This allows us to understand the implications of the market evolution in the price dynamics.

In Chapter 4 we implemented and analyzed the commodity forward curves which correspond to the storage structural model presented in Chapter 3. We also analyzed the differences between the equilibrium storage policies applied by the competitive stockholder and by the monopolist. Hence, this chapter complements and concludes the analysis of the storage structural model developed in Chapter 3. In order to obtain the commodity forward curves we developed a sophisticated

numerical procedure. This chapter presents two main contributions to the literature. First, we developed a sophisticated numerical procedure which is inspired by the one suggested by Hull and White (1993a) [55] but significantly different. This procedure can be applied to any two-state dependent stochastic dynamic control problem for which there exists a steady state policy for the controllable variable. Second, this chapter provides a comprehensive analysis of the forward curves implied by a structural model. With the exception of Routledge, Seppi and Spatt (2000) [78], the existing literature in structural models for commodity prices limits the scope of analysis to the study of the spot price properties as a function of the state variables. Hence, these papers do not analyze the forward curves implied by the corresponding models. Routledge, Seppi and Spatt (2000) [78] present a study of equilibrium forward curves conditional on the initial inventory and demand levels¹. Nevertheless, their analysis is limited to the case where the demand can only take two possible states - high and low - and it is very difficult to generalize to a more realistic number of demand states or to a more general Markov process. In contrast, the numerical procedure presented in Chapter 4 is fairly general and our analysis can easily be generalized to any combination of initial values of the supply and inventory level. Furthermore, we also provided a comparison between the forward curves observed within the competitive and the monopolistic storage markets.

Both Chapters 3 and 4 suggest testable hypothesis concerning the empirical dynamics of commodity spot/futures prices and provided us with useful insights into the most desirable properties to incorporate into reduced form models which are presented in the Chapters 5 and 6.

In Chapter 5 we developed a new reduced form model where the spot

¹As mentioned before, the source of uncertainty in their model comes from the stochastic demand.

price is driven by a mean reverting stochastic process in the absence of storage and where the possibility of storage constrains the upward drift of the spot price process. Accordingly, the commodity spot price switches between two distinct processes depending on whether or not inventory is being held. We analyzed this model in terms of both the forward curve properties and the spot price distribution properties. We illustrated and compared the properties of both our model and Schwartz (1997) [82] single-factor model using trinomial tree techniques. We showed that the new model produces a rich set of forward curves and convenience yields without violating cash-and-carry arbitrage conditions. In contrast, Schwartz model produces negative convenience yields when the initial spot price is low. Our model produces excess kurtosis in the final spot prices distribution, which is also a desirable property in a storage economy. This model contributes with an innovative approach to the literature and simultaneously obeys arbitrage-free arguments. In addition, it produces a rich set of forward curves with generally suitable properties in a storage economy. However, this model also presents two misspecifications. First, it also produces left skewness, which is inappropriate. In reality, commodity spot prices distributions are right skewed in the presence of storage since storage is more effective in stabilizing low prices than in stabilizing high prices. This is mostly due to the inventory non-negativity. Second, the shape of the forward curves in contango is not consistent with the analysis provided in Chapter 4, by Routledge, Seppi and Spatt (2000) [78] and the theory of storage. In particular, we observed that the futures returns observed in contango is smaller than the cost of carrying inventory. This result diverges from our previous analysis and from a rational behaviour since cost of holding stocks is not being covered by the futures returns when the spot price is low. These misspecifications motivated the development of a more sophisticated and realistic reduced form model in the

following chapter.

In Chapter 6 we presented a two-factor model for commodity prices and the corresponding futures valuation. This model adds important features to the literature by extending Gibson and Schwartz (1990) [42] and Schwartz (1997) [82] two-factor. First, the standard O-U process for the convenience yield is replaced by a CIR process. This allows us to maintain the mean-reverting property of the convenience yield and to additionally ensure that our model is arbitrage-free. Second, we considered that both the spot price and the convenience yield volatilities are proportional to the square root of the instantaneous convenience yield level. This specification establishes a dependency between the commodity price volatility and the inventory levels. As explained before, this property is predicted by the theory of storage and by the structural models in the literature. This characteristic is also illustrated by the structural model developed in Chapters 3 and 4. We derived the analytical solution for the futures partial differential equation. By guessing a solution of the exponential affine form, we transformed the initial PDE into a system two ODEs with analytical solution. This solution also satisfies the original PDE with the appropriate boundary conditions. This method has been applied in the interest models literature such as Hull and White (1990) [54], Brown and Shaefer (1994) [11] and Duffie and Kan (1994, 1996) [31, 32]. We empirically implemented both our model and Schwartz (1997) two-factor model for purpose of comparison using crude oil futures prices applying the Kalman filter. Although our model adds valuable characteristics to the existing reduced form models in the literature, it only outperforms Schwartz model by a small amount, in terms of both the pricing errors and the value of the log-likelihood function. Both models achieve very good results when valuating short-term maturity data but their efficiency decreases when the futures maturity increases. Neither can they re-

produce the empirical volatility structure accurately. Both models underestimate short-term volatilities and overestimate longer term volatilities. This certainly has implications on contingent claims pricing for which the volatility is a fundamental element. With hindsight, we believe that the results were affected by the peculiarity of the data set used. In particular, we do not observed contango in the whole data set available to us. One of the advantages of our model in relation to Schwartz (1997) [82] model is the exclusion of arbitrage possibilities when the forward curve is in contango. Therefore, the empirical analysis of this particular data set did not allow us to illustrate this advantage.

7.2 Limitations and Further Research

In this thesis, we developed innovative models for the price of a storable commodity within the two main streams in the literature - structural and reduced form models. The stochastic structural model presented in Chapters 3 and 4 and the two reduced form models presented in Chapter 5 and 6 represent significant theoretical advances of their type. Nevertheless, each of these presents some limitations and point out further directions for research.

The stochastic reduced form model presented and analyzed in Chapters 3 and 4 suggests three important extensions. One is the extension of the analysis to encompass non-linear demand functions, which is a more realistic assumption than having a linear demand function. Another direction of future work is the numerical implementation and analysis of the seasonal version of this model as suggested in Chapter 3. Since most of commodity prices have seasonality, the study of the seasonal effect in the price dynamics is important. It would also be useful to study the case where the supply includes jumps since one of the energy price characteristic is the occurrence of occasional spikes. This could easily be included in the model by

adding a Poisson process to the supply stochastic process in the spirit of the jump diffusion process presented by Merton (1976) [66]. Allowing capacity investment is also worth exploring. The integration of a real options model like that of Dixit and Pindyck (1994) [28] with the richer environment of this model is an interesting, and certainly challenging, possibility. Another direction is the development of a steady state version of the model presented in this chapter. Although this seems to be extremely difficult to obtain under realistic assumptions, it would be interesting and useful to find a method to study the steady state case directly.

The reduced form two-factor model provides new interesting features to the existing literature by considering time varying volatility for the spot price process and for the convenience yield process. Despite these important theoretical additions, this model only outperforms Schwartz (1997) [82] two-factor model by a small amount. We also observed that neither model is able to reproduce the futures empirical volatility structure accurately. Therefore, this study suggests that both our model and Schwartz (1997) [82] models should be extended in order to reproduce accurately the volatility structure of the commodity forward curves. This is a challenging task, since it is necessary to find a compromise between the model specification and its analytical tractability. We also believe that the empirical results presented in Chapter 6 were affected by the particularity of the data set used. Accordingly, it is worth to further explore the performance of these models by studying a different commodity or by considering a different period of time. It is also important to explore the implications of our model in option pricing and compare these implications with the existing models.

In this matter, we also suggest that further work should be carried on HJM type of models. This class of models provide a great simplification to the spot prices diffusion approach since it specifies directly the process followed by

the futures prices under the risk neutral measure. However, similarly to the spot diffusion approach, there is no constraint in the slope of the forward curve, which allows cash-and-carry opportunities. Moreover, none of the models provided by the literature fits accurately the market data. This seems to be partly due to a misspecification of the futures volatility. As a result, further research should be pursued in this direction.

Further work will also include the development of new models for particular markets. Instead of developing general models, it would be interesting to specialize the study of new models for the price of specific commodities.

Appendix A

Derivation of the Stochastic Dynamic Programming Equation

We derive the stochastic dynamic programming resulting from maximizing the following functional:

$$J(s_t, z_t, t; u(\cdot)) = E_t^z \left\{ \int_t^T e^{-r(l-t)} L(s_l, z_l, u_l, l) dl \mid + \Psi(s_T, z_T) \mid s = S, z = Z \right\} \quad (\text{A.1})$$

over all the admissible plans where the state variables s and z satisfy the following transition equations:

$$dz_t = \alpha(\bar{z} - z_t)dt + \sigma dB_t, \quad t \geq 0; \quad (\text{A.2})$$

where B_t is a standard Wiener process defined on the underlying filtered probability space $(\Omega, F, \{F_t\}_{t \geq 0}, P)$.

$$ds = u(s, z, t)dt, \quad 0 \leq s \leq b; \quad (\text{A.3})$$

and $L(s_t, z_t, u_t, t)$ is the instantaneous profit rate and $\Psi(s_T, z_T)$ is the salvage value of having s_T and z_T as states at final time T . Without loss of generality, we consider $\Psi(s_T, z_T) = 0$.

To solve the problem defined by equation (A.1), let $V(s, z, t)$, known as the *Value Function* be the expected value of the objective function in (A.1) from t to T when an optimal policy is followed from t to T , given $s_t = S$ and $z_t = Z$. Then, by the *principle of optimality*,

$$V(s, z, t) = \sup_{u \in [u_{\min}, u_{\max}]} E \{ L(s_t, z_t, u_t, t) dt + e^{-r dt} V(s + ds, z + dz, t + dt | s = S, z = Z) \} \quad (\text{A.4})$$

where $[u_{\min}, u_{\max}]$ is defined in Section 3.1.

Multiplying both sides of the equation by $e^{r dt}$ and noting that $e^{r dt} \simeq 1 + r h$ we obtain:

$$(1 + r dt) V(s, z, t) = \sup_{u \in [u_{\min}, u_{\max}]} E \{ L(s_t, z_t, u_t, t) dt + V(s + ds, z + dz, t + dt | s = S, z = Z) \} \quad (\text{A.5})$$

That is:

$$\begin{aligned} r dt V(s, z, t) &= \sup_{u \in [u_{\min}, u_{\max}]} \{ L(s_t, z_t, u_t, t) dt + E_z (V(s + ds, z + dz, t + dt) - V(s, z, t) | s = S, z = Z) \} \\ &= \sup_{u \in [u_{\min}, u_{\max}]} \left\{ L(s_t, z_t, u_t, t) dt + E_z (dV(s, z, t) | s = S, z = Z) \right\} \end{aligned} \quad (\text{A.6})$$

Applying Ito's calculus, we have:

$$dV(s, z, t) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial s}ds + \frac{\partial V}{\partial z}dz + \frac{1}{2} \frac{\partial^2 V}{\partial z^2} (dz)^2 \quad (\text{A.7})$$

where ds and dz are as above and $dz^2 = \sigma^2 dt$, which gives:

$$dV(s, z, t) = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial s}u dt + (\alpha(\bar{z} - z_t)dt + \sigma dB_t) \frac{\partial V}{\partial z} + \frac{1}{2} \sigma^2 dt \frac{\partial^2 V}{\partial z^2} \quad (\text{A.8})$$

which implies that

$$E(dV(s, z, t) | s = S, z = Z) = \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial s}u + \alpha(\bar{z} - z) \frac{\partial V}{\partial z} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial z^2} \right) dt \quad (\text{A.9})$$

Replacing (A.9) into equation (A.6) and dividing by dt gives:

$$rV(s, z, t) = \sup_{u \in [u_{\min}, u_{\max}]} \left\{ L(s_t, z_t, u_t, t) + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial s}u + \alpha(\bar{z} - z) \frac{\partial V}{\partial z} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial z^2} \right\}, \quad (\text{A.10})$$

which is the stochastic dynamic programming equation we need to solve. This equation can also be written in the following form:

$$-\frac{\partial V(s, z, t)}{\partial t} - H(s, z, V_s, V_z, V_{zz}) = 0 \quad (\text{A.11})$$

where:

$$H(s, z, V_s, V_z, V_{zz}) = \sup_{u \in [u_{\min}, u_{\max}]} \left\{ L(s_t, z_t, u_t, t) + uV_s(s, z, t) + \alpha(\bar{z} - z)V_z(s, z, t) + \frac{1}{2} \sigma^2 V_{zz}(s, z, t) - rV(s, z, t) \right\} \quad (\text{A.12})$$

Appendix B

Calculation of the Spot Prices Variability

First we point out that, because of the additive form of our model, we prefer to calculate the standard deviation of the commodity prices process dP , rather than the standard deviation of dP/P as in a conventional volatility measure.

The price function is $p(q) = a - bq$. If we write the price as a function of the two state variables in the models s and z , which are the inventory level and the supply rate and apply Ito's lemma we get:

$$dP(s, z) = P_s ds + P_z dz + \frac{1}{2} P_{zz} (dz)^2 \quad (\text{B.1})$$

where ds and dz are the transition equation of s and z respectively and are given by:

$$ds_t = u(s, z, t)dt, \quad 0 \leq s \leq b \quad (\text{B.2})$$

$$dz_t = \alpha(\bar{z} - z_t)dt + \sigma dW_t \quad (\text{B.3})$$

where W_t is a standard Wiener process. Substituting equations (B.2) and (B.3) in equation (B.1) we get:

$$dP(s, z) = (u_t P_s + \alpha(\bar{z} - z_t) + \frac{1}{2}\sigma^2 P_{zz})dt + \sigma P_z dW_t \quad (\text{B.4})$$

Ignoring the deterministic terms of the above equation, the variability of the resulting price, σ_p is given by

$$\sigma_p = \sigma P_z \quad (\text{B.5})$$

where P_z is calculated numerically as explained in Section 3.3.1.

In the absence of storage, the volatility of the spot price would be a function of the exogenous supply only, that is, $p(z)$. In the particular case of the linear function considered here we would have $p(z) = a - b(z)$, $a, b \geq 0$. Applying Ito's lemma to $p(z)$, we get:

$$dP = P_z dz_t = -b\alpha(\bar{z} - z_t)dt - b\sigma dW_t \quad (\text{B.6})$$

The variability of the price in the absence of storage is then $b\sigma$.

Appendix C

Lattice Model for the Ornstein-Uhlenbeck Process

In this section, we describe a general and efficient procedure involving the use of a trinomial tree to implement the standard O-U process given by:

$$dz_t = \alpha(\bar{z} - z_t)dt + \sigma dB_t, \quad t \geq 0 \quad (\text{C.1})$$

where:

- α is the speed of mean reversion;
- \bar{z} is the long-run mean, that is, the level to which z reverts as t goes to infinity;
- σ is the (constant) volatility;
- B_t is a standard Wiener process;

This procedure is partially based on the method described by Hull and White (1993b, 1994) [56, 57] and later revised by Clewlow and Strickland (1998) [14] and Clewlow and Strickland (2000) [16]. Originally, the method was designed to

implement short interest rate that follow a mean-reverting arithmetic stochastic process. Examples of such interest rate models are the Vasicek or Hull-White model [50], the Ho-Lee model [52] and the Black-Karasinski model [6].

The trinomial tree is constructed by using time steps of length Δt and z -steps of length Δz . At the end of each time step, z takes the value $z_0 + j\Delta z$, where j can be either positive or negative and z_0 is the initial value. (i, j) is defined as the node for which $t = i\Delta t$ and $z = j\Delta z$. The trinomial branching process can take any of the forms represented in Figure C.1. The branching process (a) is a normal branching process where we can move up by Δz , stay the same and move down by Δz . Branching process (b) occurs when $z_{i,j}$ is currently low and $z_{i,j}$ can stay the same, move up by Δz and move up by $2\Delta z$. Branching (c) occurs when $z_{i,j}$ is currently high and can stay the same, move down by Δz and move down by $2\Delta z$. In other words, the three nodes emanating from node (i, j) are $(i+1, k+1)$ - the "upper" node, $(i+1, k)$ - the "middle" node and $(i+1, k-1)$ - the "lower" node. The value of k is chosen so that $z_{i+1,k}$ is as close as possible to the expected value of z , which by definition is given by $z_{i,j} + \mu_{i,j}$, where $\mu_{i,j} = \alpha(\bar{z} + (z_0 + j\Delta z))\Delta t$. For the normal branching process $k = j$, for the branching processes illustrated by (b) and (c), $k = j + 1$ and $k = j - 1$, respectively.

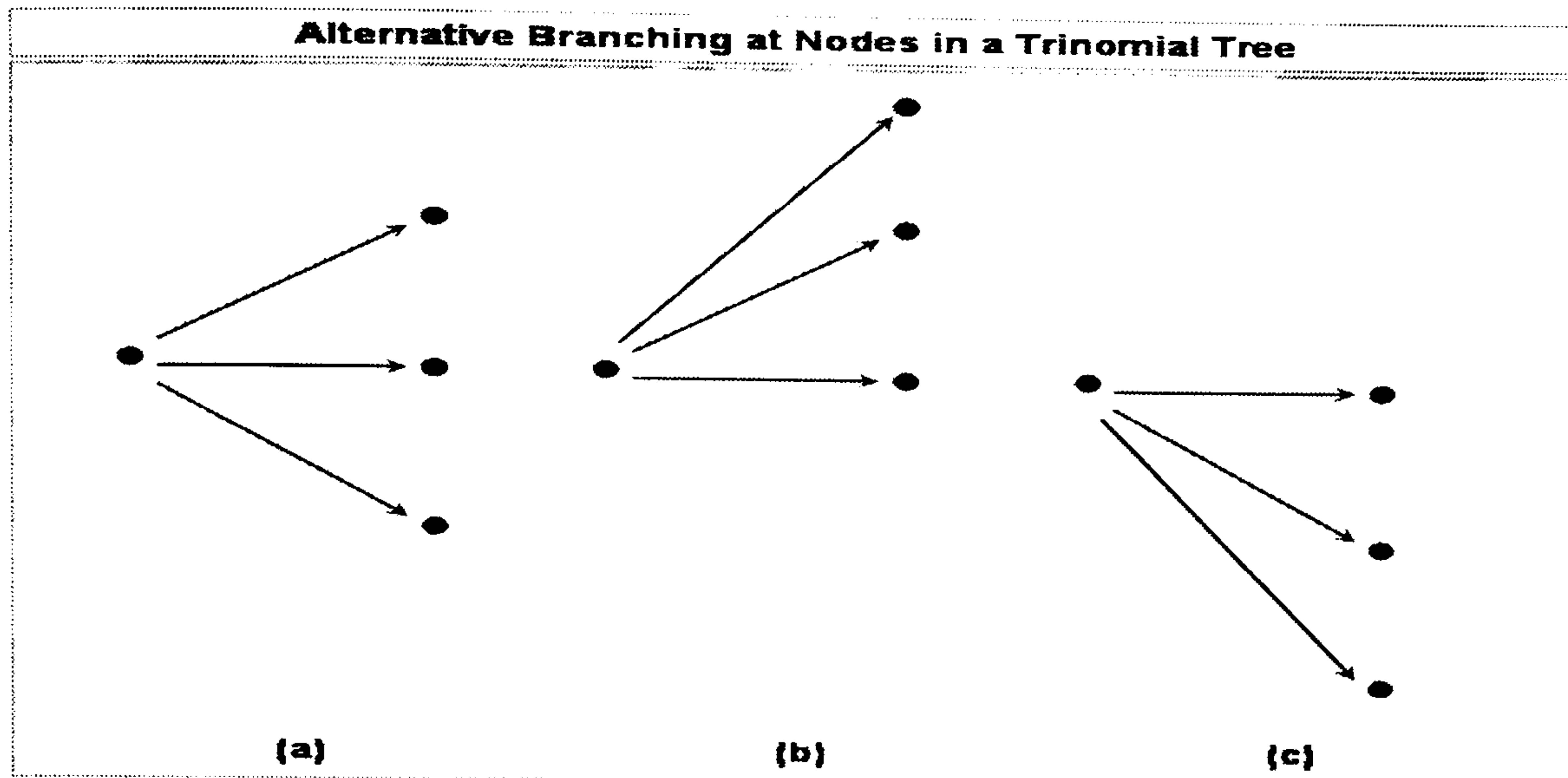


Figure C.1: Alternative Branching at Nodes in a Trinomial Tree

Given the size of the time-step, Δt , Hull and White suggest that (see Hull and White, 1990a [53]), accordingly to stability and convergence conditions, a good relationship between Δt and the space step Δz is:

$$\Delta z = \sigma \sqrt{3\Delta t} \quad (C.2)$$

(i, j) is defined as the node for which $t = i\Delta t$ and $z = j\Delta z$. Define $pr_{i,j}^u$, $pr_{i,j}^m$ and $pr_{i,j}^d$ as the probabilities of the highest, middle and lowest branches emanating from node (i, j) . The probabilities are chosen to match the expected change and variance in z over the next interval Δt . The probabilities must also sum to unity. Accordingly, the probabilities are determined by the three relationships:

$$\begin{aligned} pr_{i,j}^u + pr_{i,j}^m + pr_{i,j}^d &= 1 \\ (k+1-j)\Delta z pr_{i,j}^u + (k-j)\Delta z pr_{i,j}^m + (k-1-j)\Delta z pr_{i,j}^d &= \mu_{i,j} \\ ((k+1-j)\Delta z)^2 pr_{i,j}^u + ((k-j)\Delta z)^2 pr_{i,j}^m + (k-1-j)^2 \Delta z^2 pr_{i,j}^d &= \sigma_{i,j}^2 \Delta t + (\mu_{i,j})^2 \end{aligned} \quad (C.3)$$

where $\mu_{i,j} = E[\Delta z | (i, j)] = \alpha(\bar{z} - z_{i,j})\Delta t$ is the conditional expectation of the discretized z_t process at node (i, j) and $\sigma_{i,j}^2 \Delta t + (\mu_{i,j} \Delta t)^2 = E[\Delta z^2 | (i, j)]$, where

$\sigma_{i,j} = \sigma$ from equation (C.1) above. The probabilities are given by:

$$\begin{aligned} pr_{i,j}^u &= \frac{\sigma^2 \Delta t + \eta^2}{2\Delta z^2} + \frac{\eta}{2\Delta z} \\ pr_{i,j}^m &= 1 - \frac{\sigma^2 \Delta t + \eta^2}{\Delta z^2} \\ pr_{i,j}^d &= \frac{\sigma^2 \Delta t + \eta^2}{2\Delta z^2} - \frac{\eta}{2\Delta z} \end{aligned} \tag{C.4}$$

where $\eta = \mu_{i,j} + (j - k)\Delta z$ and $k = j - 1, j$ and $j + 1$, depending on the type of branching, as described above.

Provided that Δz is within the range $\sigma\sqrt{3\Delta t}/2$ to $2\sigma\sqrt{\Delta t}$, the probabilities are always between 0 and 1 (Hull and White, 1993b [56]).

Appendix D

Shepard Local Interpolation

This description is based on Engeln-Mullges and Uhlig (1996) [34]. To interpolate the optimal storage policy for each combination (s, z) in the tree we use the local Shepard interpolation (Shepard, 1968 [85]) with Franke-Little weights (Franke, 1982 [39]). This method has proven well suited for the graphic representation of surfaces. Moreover, we obtained more accurate results using this method instead of the Lagrange interpolation.

Its interpolating function Φ is uniquely determined independently from the ordering of the nodes (s_j, z_j) . The function $f : x = f(s, z)$, for $(s, z) \in B$, where B is an arbitrary region of the s, z plane, is interpolated for the given nodes (s_j, z_j) by the function

$$\Phi(s, z) = \sum_{j=0}^N \omega_j(s, z) f_j \quad (\text{D.1})$$

Here $\Phi(s_j, z_j) = f(s_j, z_j)$ for $j = 0, \dots, N$, where f_j are the given functional values f_j are the given functional values $f(s_j, z_j)$ at the nodes (s_j, z_j) , $j = 0, \dots, N$.

We now describe the algorithm for the Local Shepard interpolation with Franke-Little weights:

Given $N + 1$ points $(s_j, z_j, f_j = f(s_j, z_j)) \in \mathbb{R}^3$ for $j = 0, \dots, N$ with $(s_j, z_j) \in B \subseteq \mathbb{R}^2$ we find a Shepard function Φ of the form:

$$\Phi(s, z) = \sum_{j=0}^N \omega_j(s, z) f_j = \frac{\sum_{j=0}^N \left(1 - \frac{r_j(s, z)}{R}\right)^\mu f_j}{\left(1 - \frac{r_j(s, z)}{R}\right)} \quad (\text{D.2})$$

The following steps have to be carried out for each point $(s, z) \in B$ with $(s, z) = (s_j, z_j)$:

1. Choose suitable values for μ and R , usually $2 \leq \mu \leq 6$; $0.1 \leq R \leq 0.5$. According to Engeln-Mullges and Uhlig (1996) [34], it is recommended to choose a small value for R in case of many available nodes and a larger R with few known nodes. In our implementation we chose $R = 0.1$ and $\mu = 6$.
2. Calculate $r_j(s, z)$ for $j = 0, \dots, N$ where:

$$r_j(s, z) = \sqrt{(s - s_j)^2 + (z - z_j)^2}, \quad j = 0, \dots, N. \quad (\text{D.3})$$

3. Calculate the weights $\omega_j(s, z)$ for $j = 0, \dots, N$ according to:

$$\omega_j(s, z) = \frac{\xi_j^\mu(s, z)}{\sum_{i=0}^N \xi_i^\mu(s, z)} \quad (\text{D.4})$$

where

$$\xi_j^\mu(s, z) = \begin{cases} 1 - \frac{r_j(s, z)}{R}, & \text{for } 0 < r_j(s, z) < R; \\ 0, & r_j(s, z) \geq R. \end{cases} \quad (\text{D.5})$$

4. Calculate the functional value $\Phi(s, z) \approx f(s, z)$ according to equation (D.2).

Appendix E

Derivation of the Process Followed

by $x_t = \ln(p_t)$

Ito's Lemma tells us that is the process Y_t follows the diffusion $dY = \mu(\cdot)dt + \sigma(\cdot)dB_t$, where B_t represents a standard diffusion process, and $f(Y_t)$ is twice continuously differentiable, then:

$$df = \frac{\partial f}{\partial Y_t}dY + \frac{1}{2} \frac{\partial^2 f}{\partial Y_t^2}(dY)^2 \quad (\text{E.1})$$

For our problem, $f(\cdot) = \ln(\cdot)$, and thus:

$$dx = \frac{1}{p_t}dp_t - \frac{1}{2p_t^2}(dp_t)^2 \quad (\text{E.2})$$

Applying Ito's lemma to $x_t = \ln(p_t)$ where:

$$dp_t = \alpha(m - \ln(p_t))p_t dt + \sigma p_t dB_t, \quad (\text{E.3})$$

which corresponds to equation (5.1) gives:

$$dx_t = \alpha(\bar{x} - x_t)dt + \sigma dB_t \quad (\text{E.4})$$

where:

$$\bar{x} = m - \frac{\sigma^2}{2\alpha}. \quad (\text{E.5})$$

Similarly, applying Ito's lemma to the GBM described by equation:

$$dp_t = (r + c)p_t dt + \sigma p_t dB_t, \quad (\text{E.6})$$

which corresponds to equation (5.2) in Chapter 5 gives:

$$dx_t = \left((r + c) - \frac{1}{2}\sigma^2 \right) dt + \sigma dB_t \quad (\text{E.7})$$

Equations (E.4) and (E.7) correspond to equations (5.5) and (5.6).

Appendix F

Derivation of the Futures Partial Differential Equation

We derive the partial differential equation for the futures prices for the model described in Chapter 6 taking into account the martingale property of futures prices under the risk neutral measure. Specifically, the instantaneous return on all futures contracts (which require no investment) is equal to zero under the risk neutral measure (see Cox, Ingersoll and Ross, 1981 [22]). Denote $F(p, \delta, T - t = \tau)$ the commodity futures price at time t and maturity at T under the risk neutral measure. Applying Ito's lemma to $F(p, \delta, T - t = \tau)$, we obtain:

$$dF(p, \delta, \tau) = F_p dp + \frac{1}{2} F_{pp} (dp)^2 + F_\delta d\delta + \frac{1}{2} F_{\delta\delta} (d\delta)^2 + F_{p\delta} (dp d\delta) - F_\tau dt, \quad (\text{F.1})$$

where dp and $d\delta$ under the risk neutral measure are given by equations (6.6) and (6.7). Substituting these equations into the stochastic differential equation (F.1) gives:

$$\begin{aligned} dF(p, \delta, \tau) = & ((r + c - \delta)pdt + \sigma_1 \sqrt{\delta} p dB_1^*) F_p + \frac{1}{2} \sigma_1^2 \delta p^2 dt F_{pp} + \\ & ((\alpha(m - \delta) - \lambda_\delta)dt + \sigma_2 \sqrt{\delta} dB_2^*) F_\delta + \frac{1}{2} \sigma_2^2 \delta dt F_{\delta\delta} + \\ & \rho \sigma_1 \sigma_2 \delta p dt F_{p\delta} - F_\tau dt, \end{aligned} \quad (\text{F.2})$$

with boundary condition $F(p, \delta, 0) = 0$. Taking into account the martingale property of futures returns, we set the drift term of equation F.2 equal to zero, that is:

$$(r + c - \delta)pF_p + \frac{1}{2}\sigma_1^2\delta p^2 F_{pp} + (\alpha(m - \delta) - \lambda_\delta)F_\delta + \frac{1}{2}\sigma_2^2\delta F_{\delta\delta} + \rho\sigma_1\sigma_2\delta pF_{p\delta} - F_\tau = 0, \quad (\text{F.3})$$

with boundary condition $F(p, \delta, 0) = 0$. Rearranging the order of the terms of these equation we get the PDE (6.10) for the futures prices.

Appendix G

Derivation of the ODEs and corresponding solution

G.1 Derivation of the ODEs

The PDE :

$$\begin{aligned} \frac{1}{2}\sigma_1^2\delta p^2 F_{pp} + \frac{1}{2}\sigma_2^2\delta F_{\delta\delta} + \rho\sigma_1\sigma_2\delta p F_{p\delta} + (r + c - \delta)p F_p + \\ (\alpha(m - \delta) - \lambda) F_\delta - F_\tau = 0 \end{aligned} \quad (\text{G.1})$$

subject to the boundary condition $F(p, \delta, 0) = p$ suggests an exponential affine form solution:

$$F(p, \delta, \tau) = p e^{A(\tau) - B(\tau)\delta} \quad (\text{G.2})$$

with initial conditions $A(0) = 0$ and $B(0) = 0$. The partial derivatives of this function are the following:

$$\begin{aligned} F_p &= \frac{F}{p}; & F_{pp} &= 0 \\ F_\delta &= -BF; & F_{\delta\delta} &= B^2F \\ F_{p\delta} &= -\frac{BF}{p}; & F_\tau &= (A_\tau - B_\tau\delta)F \end{aligned}$$

Replacing these in the original PDE (G.1), we obtain:

$$\left(\frac{1}{2}B^2\sigma_2^2 - B\rho\sigma_1\sigma_2 - 1 + B\alpha + B_\tau \right) \delta + (r + c - B\alpha m + B\lambda - A_\tau) = 0 \quad (\text{G.3})$$

where $A(0) = 0$ and $B(0) = 0$. This generates the following system of two ODEs:

$$\frac{1}{2}\sigma_2^2 B^2 + (\alpha - \rho\sigma_1\sigma_2)B - 1 + B_\tau = 0 \quad (\text{G.4})$$

and

$$(r + c) + (\lambda - \alpha m)B - A_\tau = 0 \quad (\text{G.5})$$

with

$$A(0) = 0; \quad B(0) = 0. \quad (\text{G.6})$$

This, in turn, is also the solution to the original PDE (G.1) subject to the boundary condition $F(p, \delta, 0) = p$.

G.2 Solution to the system of differential equations (6.14) and (6.15) with initial conditions (6.16)

Because the solution to equation (6.15) depends on the solution to equation (6.14) we start by solving the latter.

Write $a_1 = \frac{1}{2}\sigma_2^2$ and $a_2 = \alpha - \rho\sigma_1\sigma_2$, then equation (6.14) becomes:

$$a_1 B^2 + a_2 B - 1 + B_\tau = 0, \quad \text{with } B(0) = 0. \quad (\text{G.7})$$

We can write this equation in the format:

$$\frac{d}{d\tau}\Phi(\tau, B) = 0 \quad \Leftrightarrow \quad M(\tau, B) + N(\tau, B)\frac{dB}{d\tau} = 0 \quad (\text{G.8})$$

if and only if there exists a function $\Phi(\tau, B)$ such that:

$$M(\tau, B) = \frac{\partial\Phi}{\partial\tau} \quad \text{and} \quad N(\tau, B) = \frac{\partial\Phi}{\partial B} \quad (\text{G.9})$$

In this case we have:

$$M(\tau, B) = a_1 B^2 + a_2 B - 1, \quad \text{and} \quad (\text{G.10})$$

$$N(\tau, B) = 1. \quad (\text{G.11})$$

The equation

$$M(\tau, B) + N(\tau, B)\frac{dB}{d\tau} = 0 \quad (\text{G.12})$$

is exact if and only if:

$$\frac{\partial M}{\partial B} = \frac{\partial N}{\partial \tau}. \quad (\text{G.13})$$

Given that we have a non-exact equation, we need to multiply both sides of the equation by the following integrating factor:

$$\mu(B) = \exp \int Q(B)dB = \frac{a_1}{a_1 B^2 + a_2 B - 1} \quad (\text{G.14})$$

where:

$$Q(B) = -\frac{2a_1 B + a_2}{a_1 B^2 + a_2 B - 1}. \quad (\text{G.15})$$

Equation (G.4) then becomes:

$$a_1 + \frac{a_1}{a_1 B^2 + a_2 B - 1} \frac{dB}{dt} = 0, \quad \text{with } B(0) = 0. \quad (\text{G.16})$$

The solution to this equation is

$$B(\tau) = \frac{2(1 - e^{-k_1 \tau})}{k_1 + k_2 + (k_1 - k_2)e^{-k_1 \tau}}, \quad (\text{G.17})$$

where

$$k_1 = \sqrt{k_2^2 + 2\sigma_2^2} \quad (\text{G.18})$$

$$k_2 = (\alpha - \rho\sigma_1\sigma_2). \quad (\text{G.19})$$

Accordingly, the solution to equation (6.15) is:

$$A(\tau) = r\tau + (\lambda - \alpha m) \int_0^\tau B(q) dq \quad (\text{G.20})$$

where:

$$\int_t^T B(q) dq = \frac{2}{k_1(k_1 + k_2)} \ln \left[\frac{(k_1 + k_2)e^{k_1 \tau} + k_1 - k_2}{2k_1} \right] + \frac{2}{k_1(k_1 - k_2)} \ln \left[\frac{k_1 + k_2 + (k_1 - k_2)e^{-k_1 \tau}}{2k_1} \right] \quad (\text{G.21})$$

where k_1 and k_2 are as before.

Appendix H

Kalman Filter Procedure

In this appendix we describe the standard Kalman filter procedure that can be found in Harvey (1989) [48] and in more generic books for computational numerical methods, such as James and Webber (2000) [58]. The description here follows the last.

The Kalman filter procedure consists in three steps. These are the prediction, the update and the parameter estimation step. At time $t - 1$ we shall have current estimates of the state variables a_{t-1} , the variance P_{t-1} of a_{t-1} , and the parameters, θ_{t-1} . (Initial estimates of a_0 and P_0 need to be supplied.).

In the prediction step we find $a_{t|t-1}$, the forecast of a_t at time $t - 1$ and $P_{t|t-1}$, the forecast of P_t at time $t - 1$. At time t we get a new observation, y_t . Here, forecasts are simply the unbiased conditional estimates:

$$a_{t|t-1} = Q_{t-1}a_{t-1} + c_t \quad (\text{H.1})$$

$$P_{t|t-1} = Q_{t-1}P_{t-1}Q'_{t-1} + R_{t-1}V_{t-1}R'_{t-1}$$

with R_{t-1} as in the transition equation (6.24), Q_{t-1} is given by (6.26) and V_{t-1} given by (6.27). In the update step we use y_t to compute estimates of a_t and P_t .

When y_t has been observed, the forecast error, v_t is:

$$v_t = y_t - Z_t a_{t|t-1} - d_t. \quad (\text{H.2})$$

The variance F_t of v_t is:

$$F_t = Z_t P_{t|t-1} Z_t' + H_t. \quad (\text{H.3})$$

with Z_t and H_t according to equation (6.23).

The new estimates a_t and P_t , in terms of v_t and F_t , are:

$$\begin{aligned} a_t &= a_{t|t-1} + P_{t|t-1} Z_t' F_t^{-1} v_t \\ P_t &= P_{t|t-1} - P_{t|t-1} Z_t' F_t^{-1} Z_t P_{t|t-1} \end{aligned} \quad (\text{H.4})$$

These are the variance minimizing conditionally unbiased estimators of a_t and P_t .

In the parameter estimation step we use a_t and P_t to compute an estimate θ_t of θ . We use the maximum quasi-likelihood¹ method to estimate parameter values by maximizing at each time step the function given by equation (6.29). As mentioned before, when the state-space is non-Gaussian, the Kalman filter can still be applied and the maximum quasi-likelihood estimator is quasi-optimal.

¹Because the prediction errors are not Gaussian for our model.

Bibliography

- [1] Kaushik Amin, Victor K. Ng, and S. Craig Pirrong. *Managing Energy Price Risk*, chapter Valuing Energy Derivatives. Risk Publications, London, 1995.
- [2] S. H. Babbs and K. B. Nowman. Kalman filtering of generalized vasicek term structure models. *Journal of Financial and Quantitative Analysis*, pages 115–130, 1999.
- [3] C. A. Ball and W. N. Tourous. Unit roots and the estimation of interest rate dynamics. *Journal of Empirical Finance*, 3:215–238, 1996.
- [4] Richard Bellman. *Dynamic Programming*. Princeton University Press, Princeton, N.J., 1957.
- [5] Hendrik Bessembinder, Jay F. Coughenour, Paul J. Seguin, and Margaret Monroe Smoller. Mean-reversion in equilibrium asset prices: Evidence from the futures term structure. *Journal of Finance*, 50(1):361–375, 1995.
- [6] F. Black and P. Karasinski. Bond and option pricing when short rates are lognormal. *Financial Analysts Journal*, July-August:52–59, 1991.
- [7] Douglas T. Breeden. Consumption risk in futures markets. *Journal of Finance*, 35:503–520, 1980.

- [8] Michael J. Brennan. The supply of storage. *American Economic Review*, 48(1):50–72, 1958.
- [9] Michael J. Brennan. *Stochastic Models and Options Models: Applications to Resources, Environment and Investment Problems*, chapter The Price of Convenience and the Valuation of Commodity Contingent Claims. North-Holland, Amsterdam, 1991.
- [10] Michael J. Brennan and Eduardo S. Schwartz. Evaluating natural resource investments. *Journal of Business*, 58(2):135:157, 1985.
- [11] R. H. Brown and S. M. Schaefer. Interest rate volatility and the shape of the term structure. *Philosophical Transactions of the Royal Society of London A*, 347:563–576, 1994.
- [12] Marcus J. Chambers and Roy E. Bailey. A theory of commodity price fluctuations. *Journal of Political Economy*, 104(5):924–957, October 1996.
- [13] Dong W. Cho and Gerald S. McDougall. The supply of storage in energy futures markets. *Journal of Futures*, 10:611–621, 1990.
- [14] Les Clewlow and Chris Strickland. *Implementing Derivatives Models*. John Wiley, London, 1998.
- [15] Les Clewlow and Chris Strickland. A multi-factor model for energy derivatives. Working Paper, School of Finance and Economics, University of Technology, Sydney, August 1999.
- [16] Les Clewlow and Chris Strickland. *Energy Derivatives: Pricing and Risk Management*. Lacima Publications, London, England, 2000.

- [17] P. H. Cootner. Returns versus speculators: Telser versus keynes. *Journal of Political Economy*, 68:396–404, 1960.
- [18] Gonzalo Cortazar and Eduardo S. Schwartz. The valuation of commodity contingent claims. Anderson Graduate School of Management, UCLA, June 1992.
- [19] Gonzalo Cortazar and Eduardo S. Schwartz. Implementing a real option model for valuing an undeveloped oil field. *International Transactions in Operational Research*, 4(2):125–137, 1997.
- [20] Gonzalo Cortazar and Eduardo S. Schwartz. Implementing a stochastic model for oil futures prices. *Energy Economics*, 25(3):215–238, 2003.
- [21] J. Cox, S. Ross, and M. Rubinstein. Option pricing: A simplified approach. *Journal of Financial Economics*, 7:229–264, 1979.
- [22] John C. Cox, Jr. Jonathan E. Ingersoll, and Stephen A. Ross. The relation between forward prices and futures prices. *Journal of Financial Economics*, 9:321–346, 1981.
- [23] John C. Cox, Jr. Jonathan E. Ingersoll, and Stephen A. Ross. A theory of the term structure of interest rates. *Econometrica*, 53(2):385–407, 1985.
- [24] Angus Deaton and Guy Laroque. On the behaviour of commodity prices. *Review of Economic Studies*, 59(198):1–23, January 1992.
- [25] Angus Deaton and Guy Laroque. Competitive storage and commodity price dynamics. *Journal of Political Economy*, 104(5):896–923, October 1996.
- [26] Richard Deaves and Itzhak Krinsky. Do futures prices for commodities embody risk premiums? *Journal of Futures Markets*, 15(6):637–648, 1995.

- [27] A. K. Dixit. *The Art of Smooth Pasting*. Chur Switzerland: Harwood Academic Publishers, 1993.
- [28] Avinash K. Dixit and Robert S. Pindyck. *Investment Under Uncertainty*. Princeton University Press, Princeton, New Jersey, 1994.
- [29] J-C. Duan and J-G. Simonato. Estimatinh and testing exponential-affine term structure models by kalman filter. Woking Paper, University of Montreal, 1995.
- [30] Darrel Duffie and Stephen Gray. *Managing Energy Price Risk*, chapter Volatility in Energy Markets. Risk Publications, London, 1995.
- [31] Darrel Duffie and Rui Kan. Multi-factor term structure models. *Philosophical Transactions of the Royal Society of London, Series A*, 347:577–586, 1993.
- [32] Darrel Duffie and Rui Kan. A yield factor model of interest rates. *Mathematical Finance*, 6:379–406, 1996.
- [33] Katherine Dusak. Future trading and investor returns: An investigation of commodity market risk premiums. *Journal of Political Economy*, 81:1387–1406, 1973.
- [34] Gisela Engeln-Mullges and Frank Uhlig. *Numerical Algorithms with C*. Springer-Verlag, 1996.
- [35] N. Epstein, P. Mayor, A. Schonbucher, E. Whalley, and P. Wilmott. The valuation of a firm advertising optimally. *The Quarterly Review of Economics and Finance*, 38(2):149–166, 1998.
- [36] Paul Fackler and Michael Livingston. Optimal storage rule. *Americal Agricultural Economics Journal*, 84(3):645–659, 2002.

- [37] Eugene F. Fama and Kenneth R. French. Commodity futures prices: Some evidence on forecast power, premiums, and the theory of storage. *Journal of Business*, 60(1):55–74, 1987.
- [38] Eugene F. Fama and Kenneth R. French. Business cycles and the behaviour of metals prices. *Journal of Finance*, 43(5):1075–1093, December 1988.
- [39] R. Franke. Scatterend data interpolation: Tests of some methods. *Mathematics of Computation*, 38:181–200, 1982.
- [40] Kenneth R. French. Detecting spot price forecasts in futures prices. *Journal of Business*, 59(2):39–54, 1986.
- [41] Alois L. H. Geyer and Stefan Pichler. A state-space approach to estimate and test multifactor cox-ingersoll-ross models of the term structute. Working Paper, May 15 1998.
- [42] Rajna Gibson and Eduardo S. Schwartz. Stochastic convenience yield and the pricing of oil contingent claims. *Journal of Finance*, 45(3):959–976, July 1990.
- [43] Robert L. Gustafson. Carryover levels for grains: A method for determining amounts that are optimal under specified conditions. *USDA Technical Bulletin* 1778, 1958a.
- [44] Robert L. Gustafson. Implications of recent research on optimal storage rules. *Journal of Farm Economics*, 40:290–300, 1958b.
- [45] F. B. Hanson and K. Naimipour. Convergence of numerical method for multistate stochastic dynamic programming. In *Proceedings of Int. Federation of Automatic Control 12th World Congress*, volume 9, pages 501–504, 1993.

- [46] Floyd B. Hanson. *Stochastic Digital Control System*, volume 76 of *Control and Dynamic Systems: Advances in Theory and Applications*, chapter Techniques in Computational Stochastic Dynamic Programming, pages 103–162. Academic Press, 1996.
- [47] Floyd B. Hanson and Dennis Ryan. Optimal harvesting with both population and price dynamics. *Mathematical BioSciences*, 148:129–146, 1998.
- [48] Andrew C. Harvey. *Forecasting, structural time series model and the Kalman filter*. Cambridge University Press, 1989.
- [49] T. B. Hasuka. Consumption betas and backwardation in commodity markets. *Journal of Finance*, 39(3):647–655, 1984.
- [50] David Heath, Robert Jarrow, and Andrew Morton. Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation. *Econometrica*, 60(1):77–105, January 1992.
- [51] Jimmy Hilliard and Jorge Reis. Valuation of commodity futures and options under stochastic convenience yields, interest rates, and jump diffusions on the spot. *Journal of Financial and Quantitative Analysis*, 33(1):61–86, 1998.
- [52] T. Ho and S. Lee. Term structure movements and pricing of interest rate claims. *Journal of Finance*, 41:1011–1029, 1986.
- [53] John Hull and Alan White. Valuing derivative securities using the explicit finite difference method. *Journal of Financial and Quantitative Analysis*, 25, 1990a.
- [54] John Hull and Alan White. Pricing interest-rate-derivative securities. *The Review of Financial Studies*, 3:573–592, 1990b.

- [55] John Hull and Alan White. Efficient procedures for valuing european and american path-dependent options. *Journal of Derivatives*, 1, 1993a.
- [56] John Hull and Alan White. One-factor interest rate models and the valuation of interest-rate derivative securities. *Journal of Financial and Quantitative Analysis*, 28(2):235–254, 1993b.
- [57] John Hull and Alan White. Numerical procedures for implementing term structure models i: Single-factor models. *Journal of Derivatives*, Fall:7–16, 1994.
- [58] Jessica James and Nick Webber. *Interest Rate Modelling*. John Wiley and Sons, 2000.
- [59] Nicholas Kaldor. Speculation and economic stability. *Review of Economic Studies*, 7(1):1–27, October 1939.
- [60] John M. Keynes. *A Treatise on Money: Volume II: The Applied Theory of Money*. Macmillan, London, 1930.
- [61] David G. Laughton and Henry D. Jacoby. Reversion, timing options, and long-term decision-making. *Financial Management*, 22(3):225–240, 1993.
- [62] Robert H. Litzenberger and Nir Rabinowitz. Backwardation in oil futures markets: Theory and empirical evidence. *The journal of Finance*, 50(3):1517–1545, December 1995.
- [63] J. Lund. Non linear kalman filter techniques for term structure models of the term structure of interest rates. Working Paper, University of Aarhus, 1997.
- [64] Robert L. Macdonald and Daniel R. Siegel. The value of waiting to invest. *Quarterly Journal of Economics*, 101:707–728, 1986.

- [65] Michaela Manoliu and Stathis Tompaidis. Energy futures prices: Term structure models with kalman filter estimation. Working Paper, MSIS Departament and Center for Computational Finance, University of Texas at Austin, November 2000.
- [66] Robert C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3:125–144, 1976.
- [67] G. E. Metcalf and K. A. Hasset. Investment under alternative return assumptions comparing random walks and mean reversion. *Journal of Economic Dynamics and Control*, 19:1471–1488, 1995.
- [68] Kristian R. Miltersen and Eduardo S. Schwartz. Pricing of options on commodity futures with stochastic term structures of convenience yields and interest rates. *Journal of Financial and Quantitative Analysis*, 33(1):33–59, March 1998.
- [69] K. W. Morton and D. F. Mayers. *Numerical Solution for Partial Differential Equations: an introduction*. Cambridge: Cambridge University Press, 1994.
- [70] John F. Muth. Rational expectations and the theory of price movements. *Econometrica*, 29:315–35, 1961.
- [71] David M. G. Newbery and Joseph E. Stiglitz. *The Theory of Commodity Price Stabilization: A Study in the Economics of Risk*. Oxford University Press, London, 1981.
- [72] Victor K. Ng and Craig Pirrong. Fundamentals and volatility: Storage, spreads and the dynamics of metal prices. *Journal of Business*, 67(2):203–230, 1994.

- [73] James L. Paddock, Daniel R. Siegel, and James L. Smith. Option valuation of claims on real assets: The case of offshore petroleum leases. *Quarterly Journal of Economics*, 103:479–508, 1988.
- [74] Dragana Pilipovic. *Energy Risk: Valuing and Managing Energy Derivatives*. McGraw-Hill, 1997.
- [75] S. Craig Pirrong. Price dynamics and derivatives prices for continuously produced, storable commodities. Working Paper, Washington University, 1998.
- [76] Haim Reisman. Movements of the term structure of commodity futures and pricing of commodity contingent claims. Working Paper, Faculty of I. E. and Management, Technion-Israel Institute of Technology, December 1991.
- [77] Jr Robert E. Lucas and Edward C. Prescott. Investment under uncertainty. *Econometrica*, 39(5):659–681, 1971.
- [78] Brian Routledge, Duane J. Seppi, and C. Spatt. Equilibrium forward curves for commodities. *The Journal of Finance*, 55(3):1297–1338, June 2000.
- [79] Paul A. Samuelson. Proof that properly anticipated prices fluctuate randomly. *Industrial Management Review*, 6(2):41–49, Spring 1965.
- [80] Paul A. Samuelson. Stochastic speculative price. *Proceedings of the National Academy of Sciences of the United States of America*, 68(2):335–337, 1971.
- [81] Jose A. Scheinkman and Jack Schechtman. A simple competitive model with production and storage. *Review of Economic Studies*, 50(3):427–441, 1983.
- [82] Eduardo S. Schwartz. The stochastic behaviour of commodity prices: Implications for valuation and hedging. *Journal of Finance*, 52(3):923–973, July 1997.

- [83] Eduardo S. Schwartz. Valuing long-term commodity assets. *Financial Management*, 27(1):57–66, 1998.
- [84] Eduardo S. Schwartz and James E. Smith. Short-term variations and long-term dynamics in commodity prices. *Management Science*, 46(7):893–911, July 2000.
- [85] D. Shepard. A two-dimensional interpolation function for irregularly spaced data. ACM National Conference, 1968.
- [86] Raul Susmel and Andrew Thompson. Volatility, storage and convenience: Evidence from natural gas markets. *Journal of Futures Markets*, 17(1):17–73, 1997.
- [87] Lester G. Telser. Futures trading and the storage of cotton and wheat. *The Journal of Political Economy*, 66(3):233–255, June 1958.
- [88] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188, 1977.
- [89] Jeffrey C. Williams and Brian D. Wright. *Storage and Commodity Markets*. Cambridge University Press, Cambridge, England, 1991.
- [90] Holbrook Working. The theory of price of storage. *American Economic Review*, 39(6):1254–1262, December 1949.
- [91] Brian D. Wright and Jeffrey C. Williams. A theory of negative prices of storage. *Journal of Futures Markets*, 3:1–13, 1989.